A family of density expansions for Lévy-type processes with default

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Abstract

We consider a defaultable asset whose risk-neutral pricing dynamics are described by an exponential Lévy-type martingale subject to default. This class of models allows for local volatility, local default intensity, and a locally dependent Lévy measure. Generalizing and extending the novel adjoint expansion technique of Pagliarani, Pascucci, and Riga (2013), we derive a family of asymptotic expansions for the transition density of the underlying as well as for European-style option prices and defaultable bond prices. For the density expansion, we also provide error bounds for the truncated asymptotic series. Additionally, for pure diffusion processes, we derive an asymptotic expansion for the implied volatility induced by European calls/puts. Our method is numerically efficient; approximate transition densities and European option prices are computed via Fourier transforms; approximate bond prices are computed as finite series. Additionally, as in Pagliarani et al. (2013), for models with Gaussian-type jumps, approximate option prices can be computed in closed form. Numerical examples confirming the effectiveness and versatility of our method are provided, as is sample Mathematica code.

Keywords: Local volatility; Lévy-type process; Asymptotic expansion; Pseudo-differential calculus; Defaultable asset

1 Introduction and literature review

A *local volatility* model is a model in which the volatility σ_t of an asset X is a function of time t and the present level of X. That is, $\sigma_t = \sigma(t, X_t)$. Among local volatility models, perhaps the most well-known is the constant elasticity of variance (CEV) model of Cox (1975). One advantage of local volatility models is that transition densities of the underlying – as well as European option prices – are often available in closed-form as infinite series of special functions (see Linetsky (2007) and references therein). Another advantage of local volatility models is that, for models whose transition density is not available in closed form, accurate density and option price approximations are readily available (see, Pagliarani and Pascucci (2011), for example). Finally, Dupire (1994) shows that one can always find a local volatility function $\sigma(t, x)$ that fits the market's implied volatility surface exactly. Thus, local volatility models are quite flexible.

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Despite the above advantages, local volatility models do suffer some shortcomings. Most notably, local volatility models do not allow for the underlying to experience jumps, the need for which is well-documented in literature (see Eraker (2004) and references therein). Recently, there has been much interest in combining local volatility models and models with jumps. Andersen and Andreasen (2000), for example, discuss extensions of the implied diffusion approach of Dupire (1994) to asset processes with Poisson jumps (i.e., jumps with finite activity). And Benhamou, Gobet, and Miri (2009) derive analytically tractable option pricing approximations for models that include local volatility and a Poisson jump process. Their approach relies on asymptotic expansions around small diffusion and small jump frequency/size limits. More recently, Pagliarani, Pascucci, and Riga (2013) consider general local volatility models with independent Lévy jumps (possibly infinite activity). Unlike, Benhamou et al. (2009), Pagliarani et al. (2013) make no small jump intensity/size assumption. Rather the authors construct an approximated solution by expanding the local volatility function as a power series. While all of the methods described in this paragraph allow for local volatility and *independent* jumps, none of these methods allow for *state-dependent* jumps.

Stochastic jump-intensity was recently identified as an important feature of equity models (see Christoffersen, Jacobs, and C (2009)). A locally dependent Lévy measure allows for this possibility. Recently, two different approaches have been taken to modeling assets with locally-dependent jump measures. Mendoza-Arriaga, Carr, and Linetsky (2010) time-change a local volatility model with a Lévy subordinator. In addition to admitting exact option-pricing formulas, the subordination technique results in a locally-dependent Lévy measure. Lorig (2012) considers another class of models that allow for state-dependent jumps. The author builds a Lévy-type processes with local volatility, local default intensity, and a local Lévy measure by considering state-dependent perturbations around a constant coefficient Lévy process. In addition to pricing formula, the author provides an exact expansion for the induced implied volatility surface.

In this paper, we consider scalar Lévy-type processes with regular coefficients, which naturally includes all the models mentioned above. Generalizing and extending the methods of Pagliarani et al. (2013), we derive a family of asymptotic expansions for the transition densities of these processes, as well as for European-style derivative prices and defaultable bond prices. The key contributions of this manuscript are as follows:

- We allow for a locally-dependent Lévy measure and local default intensity, whereas Pagliarani et al. (2013) consider a locally *independent* Lévy measure and do not allow for the possibility of default. A state-dependent Lévy measure is an important feature because it allows for incorporating local dependence into infinite activity Lévy models that have no diffusion component, such as Variance Gamma (Madan, Carr, and Chang (1998)) and CGMY/Kobol (Boyarchenko and Levendorskii (2000); Carr, Geman, Madan, and Yor (2002)).
- Unlike Benhamou et al. (2009), we make no small diffusion or small jump size/intensity assumption. Our formulae are valid for any Lévy type process with smooth and bounded coefficients, independent of the relative size of the coefficients.
- Whereas Pagliarani et al. (2013) expand the local volatility and drift functions as a Taylor series about an arbitrary point, i.e. $f(x) = \sum_n a_n(x-\bar{x})^n$, in order to achieve their approximation result, we expand the local volatility, drift, killing rate and Lévy measure in an arbitrary basis, i.e. $f(x) = \sum_n c_n B_n(x)$. This is advantageous because the Taylor series typically converges only locally, whereas other choices

of the basis functions B_n may provide global convergence in suitable functional spaces.

- Using techniques from pseudo-differential calculus, we provide explicit formulae for the Fourier transform of every term in the transition density and option-pricing expansions. In the case of state dependent Gaussian jumps the respective inverse Fourier transforms can be explicitly computed, thus providing closed form approximations for densities and prices. In the general case, the density and pricing approximations can be computed quickly and easily as inverse Fourier transforms. Additionally, when considering defaultable bonds, approximate prices are computed as a finite sum; no numerical integration is required even in the general case.
- For models with Gaussian-type jumps, we provide pointwise error estimates for transition densities. Thus, we extend the previous results of Pagliarani et al. (2013) where only the purely diffusive case is considered. Additionally, our error estimates allow for jumps with locally dependent mean, variance and intensity. Thus, for models with Gaussian-type jumps, our results also extend the results of Benhamou et al. (2009), where only the case of a constant Lévy measure is considered.
- For local volatility models with no jumps and no possibility of default, we provide an asymptotic expansion for implied volatilities corresponding to European calls/puts. As with approximate bond prices, approximate implied volatilities are computed as a finite sum of simple functions; no numerical integration is required.

The rest of this paper proceeds as follows. In Section, 2 we introduce a general class of exponential Lévy-type models with locally-dependent volatility, default intensity and Lévy measure. We also describe our modeling assumptions. Next, in Section 3, we introduce the European option-pricing problem and derive a partial integro-differential equation (PIDE) for the price of an option. In Section 4 we derive a formal asymptotic expansion (in fact, a family of asymptotic expansions) for the function that solves the option pricing PIDE (Theorem 8). Next, in Section 5, we provide rigorous error estimates for our asymptotic expansion for models with Gaussian-type jumps (Theorem 14). In Section 6 we derive an implied volatility expansion for pure diffusion models (i.e., models with no default and no jumps). This expansion involves no special functions and no integrals and can therefore be computed extremely quickly. Lastly, in Section 7, we provide numerical examples that illustrate the effectiveness and versatility of our methods. Technical proofs are provided in the Appendix. Some concluding remarks are given in Section 8.

2 General Lévy-type exponential martingales

For simplicity, we assume a frictionless market, no arbitrage, zero interest rates and no dividends. Our results can easily be extended to include locally dependent interest rates and dividends. We take, as given, an equivalent martingale measure $\mathbb{Q}$, chosen by the market on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{Q})$ satisfying the usual hypothesis of completeness and right continuity. The filtration $\mathcal{F}_t$ represents the history of the market. All stochastic processes defined below live on this probability space and all expectations are taken with respect to $\mathbb{Q}$. We consider a defaultable asset S whose risk-neutral dynamics

are given by

$$\left.\begin{aligned} S_t &= \mathbb{I}_{\{\zeta>t\}} e^{X_t}, \\ dX_t &= \mu(t,X_t)dt + \sigma(t,X_t)dW_t + \int_{\mathbb{R}} d\overline{N}_t(t,X_{t-},dz)z, \\ d\overline{N}_t(t,X_{t-},dz) &= dN_t(t,X_{t-},dz) - \nu(t,X_{t-},dz)dt, \\ \zeta &= \inf\left\{t \geq 0 : \int_0^t \gamma(s,X_s)ds \geq \mathcal{E}\right\} \end{aligned}\right\} \tag{1}$$

Here, X is a Lévy-type process with local drift function $\mu(t,x)$, local volatility function $\sigma(t,x) \geq 0$ and state-dependent Lévy measure $\nu(t,x,dz)$. We shall denote by $\mathcal{F}_t^X$ the filtration generated by X. The random variable $\mathcal{E} \sim \text{Exp}(1)$ has an exponential distribution and is independent of X. Note that ζ, which represents the default time of S, is constructed here trough the so-called *canonical construction* (see Bielecki and Rutkowski (2001)), and is the first arrival time of a doubly stochastic Poisson process with local intensity function $\gamma(t,x) \geq 0$. This way of modeling default is also considered in a local volatility setting in Carr and Linetsky (2006); Linetsky (2006), and for exponential Lévy models in Capponi et al. (2013). Since ζ is not $\mathcal{F}_t^X$-measurable we introduce the filtration $\mathcal{F}_t^D = \sigma\left(\{\zeta \leq s\}, s \leq t\right\}$ in order to keep track of the event $\{\zeta \leq t\}$. The filtration of a market observer, then, is $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^D$.

In the absence of arbitrage, S must be an $\mathcal{F}_t$-martingale. Thus, the drift $\mu(t,x)$ is fixed by $\sigma(t,x)$, $\nu(t,x,dz)$ and $\gamma(t,x)$ in order to satisfy the martingale condition [1]

$$\mu(t,x) = \gamma(t,x) - a(t,x) - \int_{\mathbb{R}} \nu(t,x,dz)(e^z - 1 - z), \qquad a(t,x) := \frac{1}{2}\sigma^2(t,x). \tag{2}$$

In order for equation (1) and subsequent computations to make sense, we will assume the following:

1. **Smoothness**. The coefficients are continuous in t and smooth in x: $a(\cdot,\cdot), \gamma(\cdot,\cdot), \nu(\cdot,\cdot,dz) \in C^{0,\infty}(\mathbb{R})$ [2].
2. **Boundedness**. The coefficients a and γ are bounded. There exists a Lévy measure

$$\bar{\nu}(dz) := \sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}} \nu(t,x,dz)$$

such that

$$\int_{\mathbb{R}} \bar{\nu}(dz)\min(1,z^2) < \infty, \qquad \int_{|z|\geq 1} \bar{\nu}(dz)e^z < \infty, \qquad \int_{|z|\geq 1} \bar{\nu}(dz)|z| < \infty. \tag{3}$$

Note that these conditions imply the existence of a positive constant $\bar{\mu} < \infty$ such that $|\mu(t,x)| < \bar{\mu}$ for all (t,x).

3. **Lipschitz Continuity**. There exists a constant C such that

$$|\mu(t,x_1) - \mu(t,x_2)|^2 + |\sigma(t,x_1) - \sigma(t,x_2)|^2 + \int_{\mathbb{R}} z^2 \left|\sqrt{\nu(t,x_1,dz)} - \sqrt{\nu(t,x_2,dz)}\right|^2 \leq C|x_1 - x_2|^2$$

for any $t \in \mathbb{R}^+$ and $x_1, x_2 \in \mathbb{R}$.

[1] We provide a derivation of the martingale condition in Section 3 Remark 2 below.

[2] This notation for ν means: $\nu(\cdot,\cdot,B) \in C^{0,\infty}(\mathbb{R})$ for any Borel set B.

4. **Existence of a density**. Either $a(\cdot,\cdot) > 0$ or for every $x \in \mathbb{R}$ there exists $\alpha(x) \in (0,2)$ such that

$$\liminf_{\varepsilon \to 0^+} \varepsilon^{\alpha(x)-2} \int_{-\varepsilon}^{\varepsilon} z^2 \nu(x, dz) > 0.$$

   These conditions guarantee that the distribution of X_t has density with respect to the Lebesgue measure (in fact, a C^∞ density) for all $t > 0$. See Sato (1999) Proposition 28.3.

Together, conditions 2 and 3 ensure that the SDE in (1) has a unique strong solution up to the default time ζ (see Oksendal and Sulem (2005), Theorem 1.19). We include Assumption 4, because, in what follows, we speak of densities. Nevertheless, the existence of the density is not strictly necessary in our analysis. Indeed, since our formulae are carried out in Fourier space, all of our computations are still formally correct even when dealing with distributions that are not absolutely continuous with respect to the Lebesgue measure.

We will relax some of these Assumptions for the numerical examples provided in Section 7. Even without the above Assumptions in force, our numerical results indicate that our approximation techniques work well.

3 Option pricing

We consider a European derivative expiring at time T with payoff $H(S_T)$ and we denote by V its no-arbitrage price. For convenience, we introduce

$$h(x) := H(e^x) \qquad \text{and} \qquad K := H(0).$$

Proposition 1. *The price V_t is given by*

$$V_t = K + \mathbb{I}_{\{\zeta > t\}} \mathbb{E}\left[e^{-\int_t^T \gamma(s, X_s) ds} \left(h(X_T) - K \right) | X_t \right], \qquad t \leq T. \tag{4}$$

The proof can be found in Section 2.2 of Linetsky (2006). Because our notation differs from that of Linetsky (2006), and because the proof is short, for the reader's convenience, we provide a derivation of Proposition 1 here.

Proof. Using risk-neutral pricing, the value V_t of the derivative at time t is given by the conditional expectation of the option payoff

$$\begin{aligned}
V_t &= \mathbb{E}\left[H(S_T) | \mathcal{F}_t \right] \\
&= \mathbb{E}\left[h(X_T) \mathbb{I}_{\{\zeta > T\}} | \mathcal{F}_t \right] + K \mathbb{E}\left[\mathbb{I}_{\{\zeta \leq T\}} | \mathcal{F}_t \right] \\
&= \mathbb{E}\left[h(X_T) \mathbb{I}_{\{\zeta > T\}} | \mathcal{F}_t \right] + K - K \mathbb{E}\left[\mathbb{I}_{\{\zeta > T\}} | \mathcal{F}_t \right] \\
&= K + \mathbb{I}_{\{\zeta > t\}} \mathbb{E}\left[e^{-\int_t^T \gamma(s, X_s) ds} \left(h(X_T) - K \right) | \mathcal{F}_t^X \right] \\
&= K + \mathbb{I}_{\{\zeta > t\}} \mathbb{E}\left[e^{-\int_t^T \gamma(s, X_s) ds} \left(h(X_T) - K \right) | X_t \right],
\end{aligned}$$

where we have used Corollary 7.3.4.2 from Jeanblanc, Yor, and Chesney (2009)to write

$$\mathbb{E}\left[(h(X_T) - K) \mathbb{I}_{\{\zeta > T\}} | \mathcal{F}_t \right] = \mathbb{I}_{\{\zeta > t\}} \mathbb{E}\left[(h(X_T) - K) e^{-\int_t^T \gamma(s, X_s) ds} | \mathcal{F}_t^X \right].$$

□

Remark 2. By Proposition 1 with $K = 0$ and $h(x) = e^x$, we have that the martingale condition $S_t = \mathbb{E}[S_T|\mathcal{F}_t]$ is equivalent to

$$\mathbb{I}_{\{\zeta>t\}} e^{X_t} = \mathbb{I}_{\{\zeta>t\}} \mathbb{E}\left[e^{-\int_t^T \gamma(s,X_s)ds + X_T} |\mathcal{F}_t\right].$$

Therefore, we see that S is a martingale if and only if the process $\exp\left(-\int_0^t \gamma(s, X_s)\, ds + X_t\right)$ is a martingale. The drift condition (2) follows by applying Itô's formula to the process $\exp\left(-\int_0^t \gamma(s, X_s)\, ds + X_t\right)$ and setting the drift term to zero.

From (4) one sees that, in order to compute the price of an option, we must evaluate functions of the form [3]

$$v(t,x) := \mathbb{E}\left[e^{-\int_t^T \gamma(s,X_s)ds} h(X_T)|X_t = x\right] = \int_{\mathbb{R}} h(y)p(t,x;T,y)dy. \tag{5}$$

Here, $p(t,x;T,y)$ is a Feynman-Kac (FK) transition density, which represents the transition density of $\log S$[4]

$$p(t,x;T,y)dy = \mathbb{Q}[\log S_T \in dy|\log S_t = x], \qquad x, y \in \mathbb{R}, \qquad 0 \le t \le T < \infty.$$

Note that the FK transition density is not a probability density since (due to the possibility that $S_T = 0$) we have

$$\int_{\mathbb{R}} p(t,x;T,y)dy \le 1.$$

It is well-known that the function $v(t,x)$, defined by (5), satisfies the Kolmogorov backward equation

$$(\partial_t + \mathcal{A})v = 0, \qquad v(T,x) = h(x). \tag{6}$$

Here, the operator $\mathcal{A}$ is the generator of the FK semigroup $\mathcal{P}_{t,T}$. That is,

$$\begin{aligned} \mathcal{P}_{t,T}h(x) &:= \mathbb{E}\left[h(X_T)e^{-\int_t^T \gamma(s,X_s)ds}|X_t = x\right], \\ \mathcal{A}f(x) &= \lim_{T\to t^+} \frac{\mathcal{P}_{t,T}f(x) - f(x)}{T-t}, \qquad \text{(if strong limit exists)} \end{aligned} \tag{7}$$

For $f \in C_0^2(\mathbb{R})$, the limit (7) exists and the operator $\mathcal{A}$ is given by

$$\begin{aligned} \mathcal{A}f(x) &= \gamma(t,x)(\partial_x f(x) - f(x)) + a(t,x)(\partial_x^2 f(x) - \partial_x f(x)) \\ &\quad - \int_{\mathbb{R}} (e^z - 1 - z)\nu(t,x,dz)\partial_x f(x) + \int_{\mathbb{R}} (f(x+z) - f(x) - z\partial_x f(x))\nu(t,x,dz). \end{aligned} \tag{8}$$

From now on we *define* $\mathrm{dom}(\mathcal{A})$ to be all functions f such that, with $\mathcal{A}$ given by (8), the derivatives and integrals in $\mathcal{A}f(x)$ exist and are finite for all $x \in \mathbb{R}$. Note also that, if one sets $h = \delta_y$, then $v(t,x)$ becomes the FK density $p(t,x;T,y)$ since

$$\int_{\mathbb{R}} \delta_y(z)p(t,x;T,z)dz = p(t,x;T,y).$$

From the analytical point of view, $p(t,x;T,y)$ represents the fundamental solution of $\partial_t + \mathcal{A}$.

[3]Note: we can accommodate stochastic interest rates and dividends of the form $r_t = r(t,X_t)$ and $q_t = q(t,X_t)$ by simply making the change: $\gamma(t,x) \to \gamma(t,x) + r(t,x)$ and $\mu(t,x) \to \mu(t,X_t) + r(t,X_t) - q(t,X_t)$.

[4]Here with $\log S$ we denote the process $X_t\mathbb{I}_{\{\zeta>t\}} - \infty\,\mathbb{I}_{\{\zeta\le t\}}$

Remark 3. If $\mathcal{G}$ is the generator of a scalar Markov process and dom$(\mathcal{G})$ contains $\mathcal{S}(\mathbb{R})$, the Schwartz space of rapidly decaying functions on $\mathbb{R}$. Then $\mathcal{G}$ must have the following form:

$$\mathcal{G}f(x) = -\gamma(x)f(x) + \mu(x)\partial_x f(x) + a(x)\partial_x^2 f(x) + \int_{\mathbb{R}} \nu(x, dz)(f(x+z) - f(x) - \mathbb{I}_{\{|z|<R\}} z\partial_x f(x)), \quad (9)$$

where $\gamma \geq 0$, $a \geq 0$, ν is a Lévy measure for every x and $R \in [0, \infty]$ (see Hoh (1998), Proposition 2.10). If one enforces on $\mathcal{G}$ the drift and integrability conditions (2) and (3), which are needed to ensure that S is a martingale, and allow setting $R = \infty$, then the operators (8) and (9) coincide (in the time-homogeneous case). Thus, the class of models we consider in this paper encompasses *all* non-negative scalar Markov martingales that satisfy the regularity and boundedness conditions of Section 2.

Remark 4. In what follows we shall systematically make use of the language of pseudo-differential calculus. More precisely, let us denote by

$$\psi_\xi(x) = \psi_x(\xi) = \frac{1}{\sqrt{2\pi}} e^{i\xi x}, \qquad x, \xi \in \mathbb{R}, \quad (10)$$

the so-called *oscillating exponential function*. Then $\mathcal{A}$ can be characterized by its action on oscillating exponential functions. Indeed, we have

$$\mathcal{A}\psi_\xi(x) = \phi(t, x, \xi)\psi_\xi(x),$$

where

$$\begin{aligned}\phi(t, x, \xi) &= \gamma(t, x)(i\xi - 1) + a(t, x)(-\xi^2 - i\xi) \\ &\quad - \int_{\mathbb{R}} \nu(t, x, dz)(e^z - 1 - z)i\xi + \int_{\mathbb{R}} \nu(t, x, dz)(e^{i\xi z} - 1 - i\xi z),\end{aligned} \quad (11)$$

is called the *symbol* of $\mathcal{A}$. Noting that

$$e^{z\partial_x} u(x) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \partial_x^n u(x) = u(x+z),$$

for any analytic function $u(x)$, we have

$$\int_{\mathbb{R}} (u(x+z) - u(x) - z\partial_x u(x))\, \nu(t, x, dz) = \int_{\mathbb{R}} \nu(t, x, dz) \left(e^{z\partial_x} - 1 - z\partial_x\right) u(x).$$

Then $\mathcal{A}$ can be represented as

$$\mathcal{A} = \phi(t, x, \mathcal{D}), \qquad \mathcal{D} = -i\partial_x,$$

since by (11)

$$\begin{aligned}\phi(t, x, \mathcal{D}) &= \gamma(t, x)(\partial_x - 1) + a(t, x)(\partial_x^2 - \partial_x) \\ &\quad - \int_{\mathbb{R}} \nu(t, x, dz)(e^z - 1 - z)\partial_x + \int_{\mathbb{R}} \nu(t, x, dz) \left(e^{z\partial_x} - 1 - z\partial_x\right).\end{aligned}$$

Moreover, if coefficients $a(t), \gamma(t), \nu(t, dz)$ are independent of x, then we have the usual characterization of $\mathcal{A}$ as a multiplication by ϕ operator in the Fourier space:

$$\mathcal{A} = \mathcal{F}^{-1} \left(\phi(t, \cdot) \mathcal{F} \right), \qquad \phi(t, \cdot) \equiv \phi(t, x, \cdot),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the (direct) Fourier and inverse Fourier transform operators respectively:

$$\mathcal{F} f(\xi) = \hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f(x) dx, \qquad \mathcal{F}^{-1} f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} f(\xi) d\xi.$$

4 Density and option price expansions (a formal description)

Our goal is to construct an approximate solution of Cauchy problem (6), which we assume has a unique classical solution. Sufficient conditions for the existence and uniqueness of solutions of second order elliptic integro-differential equations are given in Theorem II.3.1 of Garroni and Menaldi (1992). To find a solution of (6), we assume that the symbol of $\mathcal{A}$ admits an expansion of the form

$$\phi(t, x, \xi) = \sum_{n=0}^{\infty} B_n(x) \phi_n(t, \xi), \tag{12}$$

where $\phi_n(t, \xi)$ is of the form

$$\begin{aligned} \phi_n(t, \xi) &= \gamma_n(t)(i\xi - 1) + a_n(t)(-\xi^2 - i\xi) \\ &\quad - \int_{\mathbb{R}} \nu_n(t, dz)(e^z - 1 - z) i\xi + \int_{\mathbb{R}} \nu_n(t, dz)(e^{iz\xi} - 1 - iz\xi). \end{aligned} \tag{13}$$

and $\{B_n\}_{n \geq 0}$ is some expansion basis with B_n being an analytic function for each $n \geq 0$, and $B_0 = 1$. Note that $\phi_n(t, \xi)$ is the symbol of an operator

$$\mathcal{A}_n := \phi_n(t, \mathcal{D}), \qquad \mathcal{D} = -i\partial_x, \tag{14}$$

so that

$$\mathcal{A}_n \psi_\xi(x) = \phi_n(t, \xi) \psi_\xi(x). \tag{15}$$

Thus, formally the generator $\mathcal{A}$ can be written as follows

$$\mathcal{A} = \sum_{n=0}^{\infty} B_n(x) \mathcal{A}_n. \tag{16}$$

Note also that $B_n(x)$ is the symbol of the differential operator $B_n(-i\partial_\xi)$ since

$$B_n(-i\partial_\xi) \psi_x(\xi) = B_n(x) \psi_x(\xi), \qquad n \geq 0. \tag{17}$$

Typically, $\mathcal{A}_n$ is the generator of a time-dependent Lévy-type process (i.e., an *additive process*) $X^{(n)}$. In the time-independent case we have that $\phi_n(\xi) = \phi_n(t, \xi)$ is the characteristic exponent of the Lévy process $X^{(n)}$

$$\mathbb{E}\left[e^{i\xi X_t^{(n)}} \right] = e^{t\phi_n(\xi)}.$$

Example 5 (Taylor series expansion)**.** Pagliarani, Pascucci, and Riga (2013) approximate the drift and diffusion coefficients of $\mathcal{A}$ as a power series about an arbitrary point $\bar{x} \in \mathbb{R}$. In our more general setting, this corresponds to setting $B_n(x) = (x - \bar{x})^n$ and expanding the diffusion and killing coefficients $a(t, \cdot)$ and $\gamma(t, \cdot)$, as well as the Lévy measure $\nu(t, \cdot, dz)$ as follows:

$$\left.\begin{aligned} a(t,x) &= \sum_{n=0}^{\infty} a_n(t,\bar{x}) B_n(x), & a_n(t,\bar{x}) &= \frac{1}{n!}\partial_x^n a(t,\bar{x}), \\ \gamma(t,x) &= \sum_{n=0}^{\infty} \gamma_n(t,\bar{x}) B_n(x), & \gamma_n(t,\bar{x}) &= \frac{1}{n!}\partial_x^n \gamma(t,\bar{x}), \\ \nu(t,x,dz) &= \sum_{n=0}^{\infty} \nu_n(t,\bar{x},dz) B_n(x), & \nu_n(t,\bar{x},dz) &= \frac{1}{n!}\partial_x^n \nu(t,\bar{x},dz). \end{aligned}\right\} \tag{18}$$

In this case, (12) and (16) become (respectively)

$$\phi(t,x,\xi) = \sum_{n=0}^{\infty} (x-\bar{x})^n \phi_n(t,\xi), \qquad \mathcal{A} = \sum_{n=0}^{\infty} (x-\bar{x})^n \phi_n(t,\mathcal{D}),$$

where, for all $n \geq 0$, the symbol $\phi_n(t, \xi)$ is given by (13) with coefficients given by (18). The choice of $\bar{x}$ is somewhat arbitrary. However, a convenient choice that seems to work well in most applications is to choose $\bar{x}$ near X_t, the current level of X. Hereafter, to simplify notation, when discussing implementation of the Taylor-series expansion, we suppress the $\bar{x}$-dependence: $a_n(t, \bar{x}) \to a_n(t)$, $\gamma_n(t, \bar{x}) \to \gamma_n(t)$ and $\nu_n(t, \bar{x}, dz) \to \nu_n(t, dz)$.

Example 6 (Two-point Taylor series expansion)**.** Suppose f is an analytic function with domain $\mathbb{R}$ and $\bar{x}_1, \bar{x}_2 \in \mathbb{R}$. Then the *two-point Taylor series* of f is given by

$$f(x) = \sum_{n=0}^{\infty} \left(c_n(\bar{x}_1,\bar{x}_2)(x-\bar{x}_1) + c_n(\bar{x}_2,\bar{x}_1)(x-\bar{x}_2) \right) (x-\bar{x}_1)^n (x-\bar{x}_2)^n, \tag{19}$$

where

$$c_0(\bar{x}_1,\bar{x}_2) = \frac{f(\bar{x}_2)}{\bar{x}_2 - \bar{x}_1}, \qquad c_n(\bar{x}_1,\bar{x}_2) = \sum_{k=0}^{n} \frac{(k+n-1)!}{k!n!(n-k)!} \frac{(-1)^k k \partial_{\bar{x}_1}^{n-k} f(\bar{x}_1) + (-1)^{n+1} n \partial_{\bar{x}_2}^{n-k} f(\bar{x}_2)}{(\bar{x}_1 - \bar{x}_2)^{k+n+1}}. \tag{20}$$

For the derivation of this result we refer the reader to Estes and Lancaster (1966); Lopez and Temme (2002). Note truncating the two-point Taylor series expansion (19) at $n = m$ results in an expansion which of f which is of order $\mathcal{O}(x^{2m+1})$.

The advantage of using a two-point Taylor series is that, by considering the first n derivatives of a function f at two points $\bar{x}_1$ and $\bar{x}_2$, one can achieve a more accurate approximation of f over a wider range of values than if one were to approximate f using $2n$ derivatives at a single point (i.e., the usual Taylor series approximation).

If we associate expansion (19) with an expansion of the form $f(x) = \sum_{n=0}^{\infty} f_n B_n(x)$ then $f_0 B_0(x) = c_n(\bar{x}_1, \bar{x}_2)(x - \bar{x}_1) + c_n(\bar{x}_2, \bar{x}_1)(x - \bar{x}_2)$, which is *affine* in x. Thus, the terms in the two-point Taylor series expansion would not be a suitable basis in (12) since $B_0(x) \neq 1$. However, one can always introduce a constant M and define a function

$$F(x) := f(x) - M, \qquad \text{so that} \qquad f(x) = M + F(x). \tag{21}$$

Then, one can express f as

$$f(x) = M + \sum_{n=1}^{\infty} \left(C_{n-1}(\bar{x}_1, \bar{x}_2)(x - \bar{x}_1) + C_{n-1}(\bar{x}_2, \bar{x}_1)(x - \bar{x}_2) \right) (x - \bar{x}_1)^n (x - \bar{x}_2)^{n-1}, \tag{22}$$

where the C_n are as given in (20) with $f \to F$. If we associate expansion (22) with an expansion of the form $f(x) = \sum_{n=0}^{\infty} f_n B_n(x)$, then we see that $f_0 B_0(x) = M$ and one can choose $B_0(x) = 1$. Thus, as written in (22), the terms of the two-point Taylor series can be used as a suitable basis in (12).

Consider the following case: suppose $a(t,x)$, $\gamma(t,x)$ and $\nu(t,x,dz)$ are of the form

$$a(t,x) = f(x)A(t), \qquad \gamma(t,x) = f(x)\Gamma(t), \qquad \nu(t,x,dz) = f(x)\mathcal{N}(t,dz), \tag{23}$$

so that $\phi(t,x,\xi) = f(x)\Phi(t,\xi)$ with

$$\begin{aligned}\Phi(t,\xi) &= \Gamma(t)(i\xi - 1) + A(t)(-\xi^2 - i\xi) \\ &\quad - \int_{\mathbb{R}} \mathcal{N}(t,dz)(e^z - 1 - z)i\xi + \int_{\mathbb{R}} \mathcal{N}(t,dz)(e^{i\xi z} - 1 - i\xi z).\end{aligned}$$

It is certainly plausible that the symbol of $\mathcal{A}$ would have such a form since, from a modeling perspective, it makes sense that default intensity, volatility and jump-intensity would be proportional. Indeed, the Jump-to-default CEV model (JDCEV) of Carr and Linetsky (2006); Carr and Madan (2010) has a similar restriction on the form of the drift, volatility and killing coefficients.

Now, under the dynamics of (23), observe that $\phi(t,x,\xi)$ and $\mathcal{A}$ can be written as in (12) and (16) respectively with $B_0 = 1$ and

$$B_n(x) = \left(C_{n-1}(\bar{x}_1, \bar{x}_2)(x - \bar{x}_1) + C_{n-1}(\bar{x}_2, \bar{x}_1)(x - \bar{x}_2) \right) (x - \bar{x}_1)^{n-1} (x - \bar{x}_2)^{n-1}, \qquad n \geq 1. \tag{24}$$

As above C_n (capital "C") are given by (20) with $f \to F := f - M$ and

$$\phi_0(t,\xi) = M\Phi(t,\xi), \qquad \phi_n(t,\xi) = \Phi(t,\xi), \qquad n \geq 1.$$

As in example 5, the choice of $\bar{x}_1$, $\bar{x}_2$ and M is somewhat arbitrary. But, a choice that seems to work well is to set $\bar{x}_1 = X_t - \Delta$ and $\bar{x}_2 = X_t + \Delta$ where $\Delta > 0$ is a constant and $M = f(X_t)$. It is also a good idea to check that, for a given choice of $\bar{x}_1$ and $\bar{x}_2$, the two-point Taylor series expansion provides a good approximation of f in the region of interest.

Note we assumed the form (23) only for sake of simplicity. Indeed, the general case can be accommodated by suitably extending expansion (12) to the more general form

$$\phi(t,x,\xi) = \sum_{n=0}^{\infty} \sum_{i=1}^{3} B_{i,n}(x)\phi_{i,n}(t,\xi),$$

where $\phi_{i,n}$ for $i = 1,2,3$ are related to the diffusion, jump and default symbols respectively. For brevity, however, we omit the details of the general case.

Example 7 (Non-local approximation in wheighted L^2-spaces)**.** If $\{B_n\}_{n \geq 0}$ is a fixed orthonormal basis in some (possibly weighted) $L^2(\mathbb{R}, \mathfrak{m}(x)dx)$ space, then in the expansion (12) we have

$$\phi_n(t,\xi) = \langle \phi(t,\cdot,\xi), B_n(\cdot) \rangle_{\mathfrak{m}}, \qquad n \geq 0.$$

A typical example would be to choose Hermite polynomials H_n centered at $\bar{x}$ as basis functions, which (as normalized below) are orthonormal under a Gaussian weighting

$$B_n(x) = H_n(x - \bar{x}), \qquad H_n(x) := \frac{1}{\sqrt{(2n)!!\sqrt{\pi}}} \frac{\partial_x^n \exp(-x^2)}{\exp(-x^2)}, \qquad n \geq 0.$$

In this case, we have

$$\phi_n(t,\xi) = \langle \phi(t,\cdot,\xi), B_n \rangle_{\mathfrak{m}} := \int_{\mathbb{R}} \phi(t,x,\xi) B_n(x) \mathfrak{m}(x) dx, \qquad \mathfrak{m}(x) := \exp\big(-(x-\bar{x})^2 \big),$$

Once again, the choice of $\bar{x}$ is arbitrary. But, it is logical to choose $\bar{x}$ near X_t, the present level of the underlying X. In Figure 1 we compare the 2nd order two-point Taylor series, the 4th order (usual) Taylor series and the 4th order Hermite polynomial expansion for an exponential function.

Now, returning to Cauchy problem (6), we suppose that $v = v(t,x)$ can be written as follows

$$v = \sum_{n=0}^{\infty} v_n. \tag{25}$$

Following Pagliarani et al. (2013), we insert expansions (16) and (25) into Cauchy problem (6) and find

$$(\partial_t + \mathcal{A}_0) v_0 = 0, \qquad v_0(T,x) = h(x), \tag{26}$$

$$(\partial_t + \mathcal{A}_0) v_n = -\sum_{k=1}^{n} B_k(x) \mathcal{A}_k v_{n-k}, \qquad v_n(T,x) = 0. \tag{27}$$

We are now in a position to find the explicit expression for $\widehat{v}_n$, the Fourier transform of v_n in (26)-(27).

Theorem 8. *Suppose $h \in L^1(\mathbb{R}, dx)$ and let $\widehat{h}$ denote its Fourier transform. Suppose further than v_n and its Fourier transform $\widehat{v}_n$ exist. Then $\widehat{v}_n(t,\xi)$ is given by*

$$\widehat{v}_0(t,\xi) = \exp\left(\int_t^T \phi_0(s,\xi) ds \right) \widehat{h}(\xi), \tag{28}$$

$$\widehat{v}_n(t,\xi) = \sum_{k=1}^{n} \int_t^T \exp\left(\int_t^s \phi_0(u,\xi) du \right) B_k(i\partial_\xi) \phi_k(s,\xi) \widehat{v}_{n-k}(s,\xi) ds, \qquad n \geq 1. \tag{29}$$

Note that the operator $B_k(i\partial_\xi)$ acts on everything to the right of it.

Proof. See Appendix A. □

Remark 9. Assuming $\widehat{v}_n \in L^1(\mathbb{R}, dx)$, one recovers v_n using

$$v_n(t,x) = \int_{\mathbb{R}} d\xi \frac{1}{\sqrt{2\pi}} e^{i\xi x} \widehat{v}_n(t,\xi). \tag{30}$$

As previously mentioned, to obtain the FK transition densities $p(t,x;T,y)$ one simply sets $h(x) = \delta_y(x)$. In this case, $\widehat{h}(\xi)$ becomes $\psi_y(-\xi)$.

When the coefficients (a, γ, ν) are time-homogeneous, then the results of Theorem 8 simplify considerably, as we show in the following corollary.

Corollary 10 (Time-homogeneous case). *Suppose that X has time-homogeneous dynamics with the local variance, default intensity and Lévy measure given by $a(x)$, $\gamma(x)$ and $\nu(x, dz)$ respectively. Then the symbol $\phi_n(t,\xi) = \phi_n(\xi)$ is independent of t. Define*

$$\tau(t) := T - t.$$

Then, for $n \le 0$ we have

$$v_n(t,x) = u_n(\tau(t), x)$$

where

$$\widehat{u}_0(\tau,\xi) = e^{\tau\phi_0(\xi)}\widehat{h}(\xi), \tag{31}$$

$$\widehat{u}_n(\tau,\xi) = \sum_{k=1}^{n} \int_0^\tau e^{(\tau-s)\phi_0(\xi)} B_k(i\partial_\xi)\phi_k(\xi)\widehat{u}_{n-k}(s,\xi)ds, \qquad n \ge 1.$$

Proof. The proof is an algebraic computation. For brevity, we omit the details. □

Example 11. Consider the Taylor density expansion of Example 5. That is, $B_n(x) = (\bar{x} - x)^n$. Then, explicitly, $\widehat{u}_1(t,\xi)$ and $\widehat{u}_2(t,\xi)$ are given by

$$\widehat{u}_1(t,\xi) = e^{t\phi_0(\xi)}\left(t\widehat{h}(\xi)\bar{x}\phi_1(\xi) + it\phi_1(\xi)\widehat{h}'(\xi) + \frac{1}{2}it^2\widehat{h}(\xi)\phi_1(\xi)\phi_0'(\xi) + it\widehat{h}(\xi)\phi_1'(\xi)\right), \tag{32}$$

$$\begin{aligned}\widehat{u}_2(t,\xi) = e^{t\phi_0(\xi)}\bigg(&\frac{1}{2}t^2\widehat{h}(\xi)\bar{x}^2\phi_1^2(\xi) + t\widehat{h}(\xi)\bar{x}^2\phi_2(\xi) - it^2\bar{x}\phi_1^2(\xi)\widehat{h}'(\xi) - 2it\bar{x}\phi_2(\xi)\widehat{h}'(\xi)\\
&- \frac{1}{2}it^3\widehat{h}(\xi)\bar{x}\phi_1^2(\xi)\phi_0'(\xi) - it^2\widehat{h}(\xi)\bar{x}\phi_2(\xi)\phi_0'(\xi) - \frac{1}{2}t^3\phi_1(\xi)^2\widehat{h}'(\xi)\phi_0'(\xi) - t^2\phi_2(\xi)\widehat{h}'(\xi)\phi_0'(\xi)\\
&- \frac{1}{8}t^4\widehat{h}(\xi)\phi_1^2(\xi)(\phi_0'(\xi))^2 - \frac{1}{3}t^3\widehat{h}(\xi)\phi_2(\xi)(\phi_0'(\xi))^2 - \frac{3}{2}it^2\widehat{h}(\xi)\bar{x}\phi_1(\xi)\phi_1'(\xi)\\
&- \frac{3}{2}t^2\phi_1(\xi)\widehat{h}'(\xi)\phi_1'(\xi) - \frac{2}{3}t^3\widehat{h}(\xi)\phi_1(\xi)\phi_0'(\xi)\phi_1'(\xi) - \frac{1}{2}t^2\widehat{h}(\xi)(\phi_1'(\xi))^2 - 2it\widehat{h}(\xi)\bar{x}\phi_2'(\xi)\\
&- 2t\widehat{h}'(\xi)\phi_2'(\xi) - t^2\widehat{h}(\xi)\phi_0'(\xi)\phi_2'(\xi) - \frac{1}{2}t^2\phi_1(\xi)^2\widehat{h}''(\xi) - t\phi_2(\xi)\widehat{h}''(\xi) - \frac{1}{6}t^3\widehat{h}(\xi)\phi_1^2(\xi)\phi_0''(\xi)\\
&- \frac{1}{2}t^2\widehat{h}(\xi)\phi_2(\xi)\phi_0''(\xi) - \frac{1}{2}t^2\widehat{h}(\xi)\phi_1(\xi)\phi_1''(\xi) - t\widehat{h}(\xi)\phi_2''(\xi)\bigg).\end{aligned} \tag{33}$$

Higher order terms are quite long. However, they can be computed quickly and explicitly using the Mathematica code provided in Appendix B.

Remark 12. Many common payoff functions (e.g. calls and puts) are not integrable: $h \notin L^1(\mathbb{R}, dx)$. Such payoffs may sometimes be accommodated using *generalized* Fourier transforms. Assume

$$\widehat{h}(\xi) := \int_{\mathbb{R}} dx \frac{1}{\sqrt{2\pi}} e^{-i\xi x} h(x) < \infty, \qquad \text{for some } \xi = \xi_r + i\xi_i \text{ with } \xi_r, \xi_i \in \mathbb{R}.$$

Assume also that $\phi(t, x, \xi_r + i\xi_i)$ is analytic as a function of ξ_r. Then the formulas appearing in Theorem 8 and Corollary 10 are valid and integration in (30) is with respect to ξ_r (i.e., $d\xi \to d\xi_r$). For example, the payoff of a European call option with payoff function $h(x) = (e^x - e^k)^+$ has a generalized Fourier transform

$$\widehat{h}(\xi) = \int_{\mathbb{R}} dx \tfrac{1}{\sqrt{2\pi}} e^{-i\xi x}(e^x - e^k)^+ = \frac{-e^{k - ik\xi}}{\sqrt{2\pi}\,(i\xi + \xi^2)}, \qquad \xi = \xi_r + i\xi_i, \qquad \xi_r \in \mathbb{R}, \qquad \xi_i \in (-\infty, -1).$$

In any practical scenario, one can only compute a finite number of terms in (25). Thus, we define $v^{(N)}$, the Nth order approximation of v by

$$v^{(N)}(t,x) = \sum_{n=0}^{N} v_n(t,x) = \int_{\mathbb{R}} d\xi \frac{1}{\sqrt{2\pi}} e^{i\xi x} \widehat{v}^{(n)}(t,\xi), \qquad \widehat{v}^{(N)}(t,\xi) := \sum_{n=0}^{N} \widehat{v}_n(t,\xi),$$

The function $u^{(N)}(t,x)$ (which we use for time-homogeneous cases) and the approximate FK transition density $p^{(N)}(t,x;T,y)$ are defined in an analogous fashion.

Remark 13. As in Pagliarani et al. (2013), when considering models with Gaussian-type jumps, i.e., models with a state-dependent Lévy measure $\nu(t,x,dz)$ of the form (34) below, all terms in the expansion for the transition density become explicit. Likewise, for models with Gaussian-type jumps, all terms in the expansion for the price of an option are also explicit, assuming the payoff is integrable against Gaussian functions. Indeed, as we shall see in Appendix C, the main term $p^{(0)}(t,x;T,y)$ is given by a series of Gaussian densities; further approximations $p^{(N)}(t,x;T,y)$ can be represented as a suitable differential operator applied to $p^{(0)}(t,x;T,y)$ acting on the variable x (see for instance Proposition 31 for the first order).

5 Pointwise error bounds for Gaussian models

In this section we prove some pointwise error estimates for $p^{(N)}(t,x;T,y)$, the Nth order approximation of the FK density of $(\partial_t + \mathcal{A})$ with $\mathcal{A}$ as in (8). Throughout this Section, we assume Gaussian-type jumps with (t,x)-dependent mean, variance and jump intensities. Furthermore, we work specifically with the Taylor series expansion of Example 5. That is, we use basis functions $B_n(x) = (x-\bar{x})^n$.

Theorem 14. *Assume that*

$$m \le a(t,x) \le M, \qquad 0 \le \gamma(t,x) \le M, \qquad t \in [0,T], \; x \in \mathbb{R},$$

for some positive constants m and M, and that

$$\nu(t,x,dz) = \lambda(t,x)\, \mathcal{N}_{\mu(t,x),\delta^2(t,x)}(dz) := \frac{\lambda(t,x)}{\sqrt{2\pi}\delta(t,x)} e^{-\frac{(z-\mu(t,x))^2}{2\delta^2(t,x)}} dz, \tag{34}$$

with

$$m \le \delta^2(t,x) \le M, \qquad 0 \le \lambda(t,x), |\mu(t,x)| \le M, \qquad t \in [0,T], \; x \in \mathbb{R}.$$

Moreover assume that $a, \gamma, \lambda, \delta, \mu$ and their x-derivatives are bounded and Lipschitz continuous in x, and uniformly bounded with respect to $t \in [0,T]$. Let $\bar{x} = y$ in (18)*. Then, for $N \ge 1$, we have*[5]

$$\left| p(t,x;T,y) - p^{(N)}(t,x;T,y) \right| \le g_N(T-t) \left(\bar{\Gamma}(t,x;T,y) + \|\partial_x \nu\|_\infty \widetilde{\Gamma}(t,x;T,y) \right), \tag{35}$$

for any $x, y \in \mathbb{R}$ and $0 < t \le T$, where

$$g_N(s) = \mathcal{O}(s), \quad \text{as } s \to 0^+.$$

[5]Here $\|\partial_x \nu\|_\infty = \max\{\|\partial_x \lambda\|_\infty, \|\partial_x \delta\|_\infty, \|\partial_x \mu\|_\infty\}$, where $\|\cdot\|_\infty$ denotes the sup-norm on $[0,T]\times\mathbb{R}$. Note that $\|\partial_x \nu\|_\infty = 0$ if λ, δ, μ are constants.

Here, the function $\bar{\Gamma}$ is the fundamental solution of the constant coefficients jump-diffusion operator

$$\partial_t u(t,x) + \frac{\bar{M}}{2}\partial_{xx} + \bar{M}\int_{\mathbb{R}} \left(u(t,x+z) - u(t,x)\right) \mathcal{N}_{\bar{M},\bar{M}}(dz),$$

where $\bar{M}$ is a suitably large constant, and $\widetilde{\Gamma}$ is defined as

$$\widetilde{\Gamma}(t,x;T,y) = \sum_{k=0}^{\infty} \frac{\bar{M}^{k/2}(T-t)^{k/2}}{\sqrt{k!}} \mathcal{C}^{k+1}\bar{\Gamma}(t,x;T,y),$$

and where $\mathcal{C}$ being the convolution operator acting as

$$\mathcal{C}f(x) = \int_{\mathbb{R}} f(x+z)\mathcal{N}_{\bar{M},\bar{M}}(dz).$$

Proof. See Appendix C. □

Remark 15. In the proof of Theorem 14, we see that functions $\mathcal{C}^k\bar{\Gamma}$ take the following form

$$\mathcal{C}^k\bar{\Gamma}(t,x;T,y) = e^{-\bar{M}(T-t)} \sum_{n=0}^{\infty} \frac{\left(\bar{M}(T-t)\right)^n}{n!\sqrt{2\pi\bar{M}(T-t+n+k)}} \exp\left(-\frac{\left(x-y+\bar{M}(n+k)\right)^2}{2\bar{M}(T-t+n+k)}\right), \qquad k \geq 1,$$

and therefore $\widetilde{\Gamma}$ can be explicitly written as

$$\widetilde{\Gamma}(t,x;T,y) = e^{-\bar{M}(T-t)} \sum_{n,k=0}^{\infty} \frac{\left(\bar{M}(T-t)\right)^{n+\frac{k}{2}}}{n!\sqrt{k!}\sqrt{2\pi\bar{M}(T-t+n+k+1)}} \exp\left(-\frac{\left(x-y+\bar{M}(n+k+1)\right)^2}{2\bar{M}(T-t+n+k+1)}\right).$$

By Remark 15, it follows that, for any $x \neq y$ fixed, we have $\bar{\Gamma}(t,x;T,y) = \mathcal{O}(T-t)$ as $(T-t)$ tends to 0, whereas $\mathcal{C}^k\bar{\Gamma}(t,x;T,y)$, and thus also $\widetilde{\Gamma}(t,x;T,y)$, tends to a positive constant as $(T-t)$ goes to 0. It is then clear by (35) that, with $x \neq y$ fixed, the asymptotic behavior of the error, when t tends to T, changes from $(T-t)$ to $(T-t)^2$ depending on whether the Lévy measure is locally-dependent or not.

Theorem 14 extends the previous results in Pagliarani et al. (2013) where only the purely diffusive case (i.e $\lambda \equiv 0$) is considered. In that case an estimate analogous to (35) holds with

$$g_N(s) = \mathcal{O}\left(s^{\frac{N+1}{2}}\right), \quad \text{as } s \to 0^+.$$

Theorem 14 shows that for jump processes, increasing the order of the expansion for N greater than one, theoretically does not give any gain in the rate of convergence of the asymptotic expansion as $t \to T^-$; this is due to the fact that the expansion is based on a local (Taylor) approximation while the PIDE contains a non-local part. This estimate is in accord with the results in Benhamou et al. (2009) where only the case of constant Lévy measure is considered. Thus Theorem 14 extends the latter results to state dependent Gaussian jumps using a completely different technique. Extensive numerical tests showed that the first order approximation gives extremely accurate results and the precision seems to be further improved by considering higher order approximations. For example, in Figure 2 we plot the approximate transition density $p^{(N)}(t,x;T,y)$ for different values of N for the Lévy-type model considered in Section 7.1.1. We note that, for $T-t \leq 5.0$ years, the transition densities $p^{(4)}(t,x;T,y)$ and $p^{(3)}(t,x;T,y)$ are nearly identical. This is typical in our numerical experiments.

Remark 16. The proof of Theorem 14 is also interesting for theoretical purposes. Indeed, it actually represents a procedure to construct $p(t,x;T,y)$. Note that with $p^{(N)}(t,x;T,y)$ being known explicitly, Equation (35) provides pointwise estimations for the exact FK density as well.

6 Implied volatility for local volatility models

For European calls and puts, it is the implied volatility induced by an option price – rather than the option price itself – that is the quantity of primary concern. In this section we derive an implied volatility expansion for time-homogeneous local volatility models (i.e., for models with no jumps and no possibility of default).

Assumption 17. In this section only, we assume

$$S_t = e^{X_t}, \qquad dX_t = -a(X_t)dt + \sqrt{2a(X_t)}dW_t,$$

so that $\phi_0(\xi) = a_0(-\xi^2 - i\xi)$ in (12), with $a_0 > 0$.

Remark 18. Note that in the Taylor expansion of Example 5, we have $a_0 = a(\bar{x})$. In the two-point Taylor expansion of Example 6, we have $a_0 = MA$. And in the Hermite expansion of Example 7, we have $a_0 = \int_{\mathbb{R}} a(x)e^{-(x-\bar{x})^2}dx$.

Throughout this section, we fix an initial value of the underlying $X_0 = x$, a time to maturity t and a call option payoff $h(X_t) = (e^{X_t} - e^k)^+$. The price of this option $u(t,x) = u(t,x;k)$ can be computed (approximately) using Corollary 10. To simplify notation, throughout this section we will suppress all dependence on (t,x,k). However, the reader should keep in mind that option prices and their corresponding implied volatilities *do* depend on these variables. We begin our analysis with a definition of the Black-Scholes price.

Definition 19. The *Black-Scholes Price* $u^{BS} : \mathbb{R}^+ \to \mathbb{R}^+$, defined as a function of volatility σ, is given by

$$u^{BS}(\sigma) := \int d\xi\, e^{t\phi^{BS}(\xi;\sigma)}\widehat{h}(\xi)\psi_\xi(x), \qquad \phi^{BS}(\xi;\sigma) = \frac{1}{2}\sigma^2(-\xi^2 - i\xi).$$

Note that $\phi^{BS}(\cdot;\sigma)$ is simply the Lévy exponent of a Brownian motion with volatility σ and drift $-\frac{1}{2}\sigma^2$. Thus, this is simply the Fourier representation of the usual Black-Scholes price.

Remark 20. Note that, by equations (30) and (31), under Assumption 17, we have $u_0 = u^{BS}(\sqrt{2a_0})$.

Definition 21. The *implied volatility* induced by option price u is the unique positive solution σ of the equation

$$u^{BS}(\sigma) = u. \tag{36}$$

We are now in position to derive an asymptotic expansion for implied volatility. We begin by writing the option price and implied volatility as follows

$$u = \sum_{n=0}^{\infty} \varepsilon^n u_n, \qquad \varepsilon = 1, \tag{37}$$

$$\sigma = \sigma_0 + \delta^\varepsilon, \qquad \delta^\varepsilon = \sum_{n=1}^{\infty} \varepsilon^n \sigma_n,$$

We introduce the constant $\varepsilon = 1$ purely for the purposes of accounting; ε plays no role in our final result. Next, we expand u^{BS} as a power series about the point σ_0. We have

$$\begin{aligned}
u^{BS}(\sigma) = u^{BS}(\sigma_0 + \delta^\varepsilon) &= \sum_{n=0}^{\infty} \frac{1}{n!}(\delta^\varepsilon \partial_\sigma)^n u^{BS}(\sigma_0) = u^{BS}(\sigma_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{k=1}^{\infty} \varepsilon^k \sigma_k \right)^n \partial_\sigma^n u^{BS}(\sigma_0) \\
&= u^{BS}(\sigma_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\sum_{k=n}^{\infty} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \varepsilon^k \right] \partial_\sigma^n u^{BS}(\sigma_0) \\
&= u^{BS}(\sigma_0) + \sum_{k=1}^{\infty} \varepsilon^k \left[\sum_{n=1}^{k} \frac{1}{n!} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \partial_\sigma^n \right] u^{BS}(\sigma_0) \\
&= u^{BS}(\sigma_0) + \sum_{k=1}^{\infty} \varepsilon^k \left[\sigma_k \partial_\sigma + \sum_{n=2}^{k} \frac{1}{n!} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \partial_\sigma^n \right] u^{BS}(\sigma_0).
\end{aligned} \tag{38}$$

Now, we insert expansions (37) and (38) into (36) and collect terms of like order in ε

$$\begin{aligned}
\mathcal{O}(1): &\qquad u_0 = u^{BS}(\sigma_0), \\
\mathcal{O}(\varepsilon^k): &\qquad u_k = \sigma_k \partial_\sigma u^{BS}(\sigma_0) + \sum_{n=2}^{k} \frac{1}{n!} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \partial_\sigma^n u^{BS}(\sigma_0), \qquad k \geq 1.
\end{aligned}$$

Solving the above equations for $\{\sigma_k\}_{k=0}^{\infty}$ we find

$$\mathcal{O}(1): \qquad \sigma_0 = \sqrt{2a_0}, \tag{39}$$

$$\mathcal{O}(\varepsilon^k): \qquad \sigma_k = \frac{1}{\partial_\sigma u^{BS}(\sigma_0)} \left(u_k - \sum_{n=2}^{k} \frac{1}{n!} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \partial_\sigma^n u^{BS}(\sigma_0) \right), \qquad k \geq 1, \tag{40}$$

where we have used $u_0 = u^{BS}(\sqrt{2a_0})$ to deduce that $\sigma_0 = \sqrt{2a_0}$ (see Remark 20). Observe, the right hand side of (40) involves only σ_j for $j \leq k-1$. Thus, the $\{\sigma_k\}_{k=1}^{\infty}$ can be found recursively. Explicitly, up to $\mathcal{O}(\varepsilon^4)$ we have

$$\left.\begin{aligned}
\mathcal{O}(\varepsilon): &\quad \sigma_1 = \frac{u_1}{\partial_\sigma u_0}, \\
\mathcal{O}(\varepsilon^2): &\quad \sigma_2 = \frac{u_2 - \frac{1}{2!}\sigma_1^2 \partial_\sigma^2 u_0}{\partial_\sigma u_0}, \\
\mathcal{O}(\varepsilon^3): &\quad \sigma_3 = \frac{u_3 - (\sigma_2 \sigma_1 \partial_\sigma^2 + \frac{1}{3!}\sigma_1^3 \partial_\sigma^3) u_0}{\partial_\sigma u_0}, \\
\mathcal{O}(\varepsilon^4): &\quad \sigma_4 = \frac{u_4 - (\sigma_3 \sigma_1 \partial_\sigma^2 + \frac{1}{2}\sigma_2^2 \partial_\sigma^2 + \frac{1}{2}\sigma_2 \sigma_1^2 \partial_\sigma^3 + \frac{1}{24}\sigma_1^4 \partial_\sigma^4) u_0}{\partial_\sigma u_0},
\end{aligned}\right\} \tag{41}$$

where

$$\partial_\sigma^n u_0 = \partial_\sigma^n u^{BS}(\sigma)|_{\sigma=\sqrt{2a_0}}.$$

Theoretically, one can use recursion relation (40) ad infinitum to find σ_n for arbitrarily large n. To sum up, our nth order asymptotic approximation for implied volatility is as follows:

$$\sigma^{(n)} = \sum_{k=0}^{n} \sigma_k,$$

where the $\{\sigma_k\}$ are given by (39) and (40).

As written, equations (41) are not very convenient. Indeed, for each u_n and for all terms of the form $\partial_\sigma^n u_0$, one must be compute Fourier integrals. However, with a little more work, we can compute the $\{\sigma_i\}$ explicitly with no integrals and no special functions. The first step is to express u_n as

$$u_n = \mathcal{L} u_0, \qquad \mathcal{L} = \sum_{k \geq 0} b_k(t,x) \partial_x^k (\partial_x^2 - \partial_x),$$

where the $\{b_k\}$ are functions of (t,x).

Assumption 22. For the remainder of Section 6, we assume $B_n(x)$ and $\phi_n(\xi) \equiv \phi_n(t,\xi)$ as in Example 5. That is, $B_n(x) = (x - \bar{x})^n$ and $\phi_n(\xi) = a_n(-\xi^2 - i\xi)$ with $a_n = \frac{1}{n!}\partial_x^n a(\bar{x})$. This choice is not necessary, but simplifies the computations that follow. Similar implied volatility results can be derived for other choices of $B_n(x)$ and $\phi_n(\xi)$.

As shown in Appendix D, under Assumption 22, for u_1 we have

$$u_1(t,x) = \Big(t(x-\bar{x}) + \tfrac{1}{2} t^2 a_0 (2\partial_x - 1) \Big) a_1 (\partial_x^2 - \partial_x) u_0(t,x). \tag{42}$$

and for u_2 we have

$$\begin{aligned} u_2(t,x) = &\Big(t(x-\bar{x})^2 + t^2 (x-\bar{x}) a_0 (2\partial_x - 1) + \tfrac{1}{3} t^3 a_0^2 (2\partial_x - 1)^2 + t^2 a_0 \Big) a_2 (\partial_x^2 - \partial_x) u_0(t,x) \\ &+ \Big(\tfrac{1}{2} t^2 (x-\bar{x})^2 a_1 (\partial_x^2 - \partial_x) + \tfrac{1}{2} t^3 \, (x-\bar{x}) \, a_1 (\partial_x^2 - \partial_x) a_0 (2\partial_x - 1) \\ &\quad + \tfrac{1}{8} t^4 a_1 (\partial_x^2 - \partial_x) a_0^2 (2\partial_x - 1)^2 + \tfrac{1}{2} t^2 \, (x-\bar{x}) \, a_1 (2\partial_x - 1) \\ &\quad + \tfrac{1}{6} t^3 a_0 (2\partial_x - 1) a_1 (2\partial_x - 1) + \tfrac{1}{3} t^3 a_1 (\partial_x^2 - \partial_x) a_0 \Big) a_1 (\partial_x^2 - \partial_x) u_0(t,x). \end{aligned} \tag{43}$$

Now, using equations (42)-(43), as well as

$$\partial_\sigma u^{BS}(\sigma) = t\sigma(\partial_x^2 - \partial_x) u^{BS}(\sigma), \qquad \frac{\partial_x^m (\partial_x^2 - \partial_x) u^{BS}(\sigma)}{(\partial_x^2 - \partial_x) u^{BS}(\sigma)} = \frac{\partial_x^m \exp\left(x - \frac{d_+^2}{2}\right)}{\exp\left(x - \frac{d_+^2}{2}\right)},$$

where $d_+ = \frac{1}{\sigma\sqrt{t}}\left(x - k + \frac{1}{2} t \, \sigma^2\right)$, a straightforward (but tedious) algebraic computation shows that

$$\begin{aligned} \sigma_1 &= \frac{u_1}{\partial_\sigma u_0} = \frac{a_1}{\sqrt{2a_0}} \left(\frac{1}{2}(x+k) - \bar{x} \right), \\ \sigma_2 &= \frac{u_2 - \frac{1}{2!}\sigma_1^2 \partial_\sigma^2 u_0}{\partial_\sigma u_0} \\ &= \frac{a_2}{3\sqrt{2a_0}} \Big(k^2 + kx + x^2 + t a_0 - 3(k+x)\bar{x} + 3\bar{x}^2 \Big) \\ &\qquad - \frac{a_1^2}{48 a_0 \sqrt{2a_0}} \Big(t a_0 (6 + t a_0) + 6\left(k^2 + x^2 - 2(k+x)\bar{x} + 2\bar{x}^2\right) \Big). \end{aligned}$$

If we set $\bar{x} \to x$ then we have the following second order approximation for implied volatility

$$\begin{aligned} \sigma^{(2)} &= \sigma_0 + \sigma_1 + \sigma_2 \\ &= \sqrt{2a_0} + \frac{a_1}{2\sqrt{2a_0}}(k-x) + \frac{a_2}{3\sqrt{2a_0}}\Big((k-x)^2 + t a_0 \Big) - \frac{a_1^2}{48 a_0 \sqrt{2a_0}} \Big(6(k-x)^2 + t a_0 \, (6 + t a_0) \Big), \end{aligned} \tag{44}$$

which is quadratic in $(k-x)$ and in t.

7 Examples

In this section, in order to illustrate the versatility of our asymptotic expansion, we apply our approximation technique to a variety of different Lévy-type models. We consider both finite and infinite activity Lévy-type measures, models with and without a diffusion component, and models with and without jumps. We study not only option prices, but also implied volatilities and credit spreads. In each setting, if the exact or approximate option price/implied volatility/credit spread has been computed by a method other than our own, we compare this to the option price/implied volatility/credit spread obtained by our approximation. For cases where the exact or approximate density/option price is not analytically available, we use Monte Carlo methods to verify the accuracy of our method.

Note that, some of the examples considered below do not satisfy the conditions listed in Section 2. In particular, we will consider coefficients (a, γ, ν) that are not bounded. Nevertheless, the formal results of Section 4 work well in the examples considered.

7.1 CEV-like Lévy-type processes

We consider a Lévy-type process of the form (1) with CEV-like volatility and jump-intensity. Specifically, the log-price dynamics are given by

$$a(x) = \frac{1}{2}\delta^2 e^{2(\beta-1)x}, \qquad \nu(x, dz) = e^{2(\beta-1)x}\mathcal{N}(dz), \qquad \gamma(x) = 0, \qquad \delta \geq 0, \qquad \beta \in [0, 1], \quad (45)$$

where $\mathcal{N}(dx)$ is a Lévy measure. When $\mathcal{N} \equiv 0$, this model reduces to the CEV model of Cox (1975). Note that, with $\beta \in [0, 1)$, the volatility and jump-intensity increase as $x \to -\infty$, consistent with the leverage effect (i.e., a decrease in the value of the underlying is often accompanied by an increase in volatility/jump intensity). This characterization will yield a negative skew in the induced implied volatility surface. As the class of models described by (45) is of the form (23) with $f(x) = e^{2(\beta-1)x}$, this class naturally lends itself to the two-point Taylor series approximation of Example 6. Thus, for certain numerical examples in this Section, we use basis functions B_n given by (24). In this case we choose expansion points $\bar{x}_1$ and $\bar{x}_2$ in a symmetric interval around X_0 and in (21) we choose $M = f(X_0) = e^{2(\beta-1)X_0}$. For other numerical examples, we use the (usual) one-point Taylor series expansion $B_n(x) = (x - \bar{x})^n$. In this cases, we choose $\bar{x} = X_0$.

We will consider two different characterizations of $\mathcal{N}(dz)$:

$$\text{Gaussian:} \qquad \mathcal{N}(dz) = \lambda \frac{1}{\sqrt{2\pi s^2}} \exp\left(\frac{-(z-m)^2}{2s^2}\right) dz, \quad (46)$$

$$\text{Variance-Gamma:} \qquad \mathcal{N}(dz) = \left(\frac{e^{-\lambda_-|z|}}{\kappa|z|}\mathbb{I}_{\{z<0\}} + \frac{e^{-\lambda_+ z}}{\kappa z}\mathbb{I}_{\{z>0\}}\right) dz, \quad (47)$$

$$\lambda_\pm = \left(\sqrt{\frac{\theta^2\kappa^2}{4} + \frac{\rho^2\kappa}{2}} \pm \frac{\theta\kappa}{2}\right)^{-1}$$

Note that the Gaussian measure is an example of a finite-activity Lévy measure (i.e., $\mathcal{N}(\mathbb{R}) < \infty$), whereas the Variance-Gamma measure, due to Madan et al. (1998), is an infinite-activity Lévy measure (i.e., $\mathcal{N}(\mathbb{R}) = \infty$). As far as the authors of this paper are aware, there is no closed-form expression for option prices (or the

transition density) in the setting of (45), regardless of the choice of $\mathcal{N}(dz)$. As such, we will compare our pricing approximation to prices of options computed via standard Monte Carlo methods.

7.1.1 Gaussian Lévy Measure

In our first numerical experiment, we consider the case of Gaussian jumps. That is, $\mathcal{N}(dz)$ is given by (46). We fix the following parameters

$$\delta = 0.20, \qquad \beta = 0.25, \qquad \lambda = 0.3, \qquad m = -0.1, \qquad s = 0.4, \qquad S_0 = e^x = 1.$$

Using Corollary 10, we compute the approximate prices $u^{(0)}(t, x; K)$ and $u^{(3)}(t, x; K)$ of a series of European puts over a range of strikes K and with times to maturity $t = \{0.25, 1.00, 3.00, 5.00\}$ (we add the parameter K to the arguments of $u^{(n)}$ to emphasize the dependence of $u^{(n)}$ on the strike price K). To compute $u^{(i)}(t, x; K)$, $i = \{0, 3\}$ we use the we the usual one-point Taylor series expansion (Example 5). We also compute the price $u(t, x; K)$ of each put by Monte Carlo simulation. For the Monte Carlo simulation, we use a standard Euler scheme with a time-step of 10^{-3} years, and we simulate 10^6 sample paths. We denote by $u^{(MC)}(t, x; K)$ the price of a put obtained by Monte Carlo simulation. As prices are often quoted in implied volatilities, we convert prices to implied volatilities by inverting the Black-Scholes formula numerically. That is, for a given put price $u(t, x; K)$, we find $\sigma(t, K)$ such that

$$u(t, x; K) = u^{BS}(t, x; K, \sigma(t, K)), \tag{48}$$

where $u^{BS}(t, x; K, \sigma)$ is the Black-Scholes price of the put as computed assuming a Black-Scholes volatility of σ. For convenience, we introduce the notation

$$\mathrm{IV}[u(t, x; K)] := \sigma(t, K)$$

to indicate the implied volatility induced by option price $u(t, x; K)$. The results of our numerical experiments are plotted in Figure 3. We observe that $\mathrm{IV}[u^{(3)}(t, x; K)]$ agrees almost exactly with $\mathrm{IV}[u^{(MC)}(t, x; K)]$. The computed prices $u^{(3)}(t, x; K)$ and their induced implied volatilities $\mathrm{IV}[u^{(3)}(t, x; K)]$, as well as 95% confidence intervals resulting from the Monte Carlo simulations can be found in Table 1.

7.1.2 Variance Gamma Lévy Measure

In our second numerical experiment, we consider the case of Variance Gamma jumps. That is, $\mathcal{N}(dz)$ given by (47). We fix the following parameters:

$$\delta = 0.0, \qquad \beta = 0.25, \qquad \theta = -0.3, \qquad \rho = 0.3, \qquad \kappa = 0.15, \qquad S_0 = e^x = 1.$$

Note that, by letting $\delta = 0$, we have set the diffusion component of X to zero: $a(x) = 0$. Thus, X is a pure-jump Lévy-type process. Using Corollary 10, we compute the approximate prices $u^{(0)}(t, x; K)$ and $u^{(2)}(t, x; K)$ of a series of European puts over a range of strikes and with maturities $t \in \{0.5, 1.0\}$. To compute $u^{(i)}$, $i \in \{0, 2\}$, we use the two-point Taylor series expansion (Example 6). We also compute the

put prices by Monte Carlo simulation. For the Monte Carlo simulation, we use a time-step of 10^{-3} years and we simulate 10^6 sample paths. At each time-step, we update X using the following algorithm

$$
\begin{aligned}
X_{t+\Delta t} &= X_t + b(X_t)\Delta t + \gamma^+(X_t) - \gamma^-(X_t), & I(x) &= e^{2(\beta-1)x}, \\
b(x) &= -\frac{I(x)}{\kappa}\left(\log\left(\frac{\lambda_-}{1+\lambda_-}\right) + \log\left(\frac{\lambda_+}{\lambda_+ - 1}\right)\right), & \gamma^\pm(x) &\sim \Gamma(I(x)\cdot\Delta t/\kappa, 1/\lambda_\pm),
\end{aligned}
$$

where $\Gamma(a,b)$ is a Gamma-distributed random variable with shape parameter a and scale parameter b. Note that this is equivalent to considering a VG-type process with state-dependent parameters

$$
\kappa'(x) := \kappa/I(x), \qquad \theta'(x) := \theta I(x), \qquad \rho'(x) := \rho\sqrt{I(x)}.
$$

These state-dependent parameters result in state-*independent* $\lambda_\pm$ (i.e., $\lambda_\pm$ remain constant). Once again, since implied volatilities rather than prices are the quantity of primary interest, we convert prices to implied volatilities by inverting the Black-Scholes formula numerically. The results are plotted in Figure 4. We observe that $\mathrm{IV}[u^{(2)}(t,x;K)]$ agrees almost exactly with $\mathrm{IV}[u^{(MC)}(t,x;K)]$. Values for $u^{(2)}(t,x;K)$, the associated implied volatilities $\mathrm{IV}[u^{(2)}(t,x;K)]$ and the 95% confidence intervals resulting from the Monte Carlo simulation can be found in table 2.

7.2 Comparison with Lorig (2012)

In Lorig (2012), the author considers a class of time-homogeneous Lévy-type processes of the form:

$$
\left.
\begin{aligned}
a(x) &= \frac{1}{2}\Big(b_0^2 + \varepsilon b_1^2 \eta(x)\Big), \\
\gamma(x) &= c_0 + \varepsilon c_1 \eta(x), \\
\nu(x,dz) &= \nu_0(dz) + \varepsilon\eta(x)\nu_1(dz).
\end{aligned}
\right\} \tag{49}
$$

Here, $(b_0, b_1, c_0, c_1, \varepsilon)$ are non-negative constants, the function $\eta \geq 0$ is smooth and ν_0 and ν_1 are Lévy measures. When $\eta(x) = e_\beta(x) := e^{\beta x}$, the author obtains the following expression for European-style options written on X

$$
u(t,x) = \sum_{n=0}^{\infty} \varepsilon^n w_n(t,x), \tag{50}
$$

$$
w_n(t,x) = e_{n\beta}(x)\int_{\mathbb{R}} d\xi \left(\sum_{k=0}^{n} \frac{e^{t\pi_{\xi-ik\beta}}}{\prod_{j\neq k}^{n}(\pi_{\xi-ik\beta} - \pi_{\xi-ij\beta})}\right)\left(\prod_{k=0}^{n-1}\chi_{\xi-ik\beta}\right)\widehat{h}(\xi)\psi_\xi(x).
$$

where t is the time to maturity, x is the present value of X and

$$
\begin{aligned}
\pi_\xi &= \frac{1}{2}b_0^2\left(-\xi^2 - i\xi\right) + c_0(i\xi - 1) - \int_{\mathbb{R}} \nu_0(dz)\Big(e^z - 1 - z\Big)i\xi + \int_{\mathbb{R}} \nu_0(dz)\Big(e^{i\xi z} - 1 - i\xi z\Big), \\
\chi_\xi &= \frac{1}{2}b_1^2\left(-\xi^2 - i\xi\right) + c_1(i\xi - 1) - \int_{\mathbb{R}} \nu_1(dz)\Big(e^z - 1 - z\Big)i\xi + \int_{\mathbb{R}} \nu_1(dz)\Big(e^{i\xi z} - 1 - i\xi z\Big).
\end{aligned}
$$

As in Corollary 10, $\widehat{h}(\xi)$ is the (possibly generalized) Fourier transform of the option payoff $h(x)$, and ψ_ξ is as given in (10).

Now consider the following model:

$$a(x) = Af(x), \qquad \gamma(x) = \Gamma f(x), \qquad \nu(x, dz) = f(x)\mathcal{N}(dz) \qquad f(x) = a_0 + \varepsilon a_1 \eta(x), \tag{51}$$

The models described by (49) and (51) coincide if we choose

$$\frac{1}{2}b_i^2 = a_i A, \qquad c_i = a_i \Gamma, \qquad \nu_i(dz) = a_i \mathcal{N}(dz), \qquad i = \{0, 1\}.$$

Furthermore, comparing equations (23) with (51), we see that (51) is precisely the form considered in Example 6. Thus, in this Section we use the two-point Taylor series approximation of Example 6 with basis functions B_n given by (24). We choose expansion points $\bar{x}_1$ and $\bar{x}_2$ in a symmetric interval around X_0 and in (21) we choose $M = f(X_0) = e^{\beta X_0}$.

In our numerical experiment, we consider Gaussian jumps (i.e., $\mathcal{N}(dz)$ given by (46)) and we fix the following parameters:

$$\varepsilon = a_0 = a_1 = 1, \qquad \beta = -2.0, \qquad A = \frac{1}{2}0.15^2, \qquad \Gamma = 0.0, \qquad \lambda = s = 0.2,$$
$$m = -0.2, \qquad t = 0.5, \qquad X_0 = 0.0, \qquad \bar{x}_1 = -0.3, \qquad \bar{x}_2 = 0.3.$$

Using Corollary 10, we compute the approximate prices $u^{(0)}(t, x; K)$ and $u^{(2)}(t, x; K)$ of a series of European puts with strike prices $K \in [0.5, 1.5]$ (once again, we add the parameter K to the arguments of $u^{(n)}$ to emphasize the dependence of $u^{(n)}$ on the strike price K). We also compute the price $u(t, x; K)$ using (50). In (50), we truncate the infinite sum at $n = 8$. As in Section 7.1, we convert option prices to implied volatilities. The results are plotted in Figure 5. We observe a nearly exact match between the induced implied volatilities $\mathrm{IV}[u^{(2)}(t, x; K)]$ and $\mathrm{IV}[u(t, x; K)]$, where $u(t, x; K)$ (with no superscript) is computed by truncating (50) at $n = 8$.

7.3 Comparison to NIG-type processes

A *Normal Inverse Gaussian* (NIG) (see Barndorff-Nielsen (1998)) is a Lévy process with characteristic Lévy exponent $\phi(\xi)$ given by

$$\phi(\xi) = i\mu\xi - \delta\left[\sqrt{\alpha^2 - (\beta + i\xi)^2} - \sqrt{\alpha^2 - \beta^2}\right].$$

In Chapter 14, equation 14.1 of Boyarchenko and Levendorskii (2000), that authors consider NIG-like Feller processes with symbol

$$\phi(x, \xi) = i\mu(x)\xi - \delta(x)\left[\sqrt{\alpha^2(x) - (\beta(x) + i\xi)^2} - \sqrt{\alpha^2(x) - \beta^2(x)}\right],$$

where $\mu, \delta, \alpha, \beta \in C_b^\infty(\mathbb{R})$, $\delta, \alpha > 0$, $\mu, \beta \in \mathbb{R}$, and where there exist constants c and C such that $\delta(x) > c$, $\alpha(x) - |\beta(x)| > c$ and $|\mu(x)| \leq C$. Note that if X is a NIG-type process with symbol $\phi(x, \xi)$, then $S = e^X$ is a martingale if and only if $\phi(x, -i) = 0$. Thus, the triple (α, β, δ) fixes μ.

Boyarchenko and Levendorskii (2000) deduce the following asymptotic expansion for $u(t, x)$ (see the equations following (14.27) and equation (16.40)).

$$u(t, x) := \mathbb{E}_x h(X_t) = \int_{\mathbb{R}} d\xi \frac{1}{\sqrt{2\pi}} e^{i\xi x} e^{t\phi(x,\xi)} \left(1 + \frac{1}{2}t^2[i\partial_x \phi(x, \xi)][\partial_\xi \phi(x, \xi)] + \cdots\right) \widehat{h}(\xi), \tag{52}$$

We note that, if one chooses basis functions $B_n(x) = (x-\bar{x})^n$ as in Example 5 with $\bar{x} = x$, then $\phi(x, \xi) = \phi_0(\xi)$ and $\partial_x \phi(x, \xi) \partial_\xi \phi(x, \xi) = \phi_1(\xi)\phi_0'(\xi)$. Thus, from Corollary 10 and equations (31) and (32), it is easy to see that expansion (52) is contained within the first two terms of the (one-point) Taylor expansion obtained in this paper.

7.4 Yields and credit spreads in the JDCEV setting

Consider a defautable bond, written on S, that pays one dollar at time $T > t$ if no default occurs prior to maturity (i.e., $S_T > 0$, $\zeta > T$) and pays zero dollars otherwise. Then the time t value of the bond is given by

$$V_t = \mathbb{E}[\mathbb{I}_{\{\zeta > T\}} | \mathcal{F}_t] = \mathbb{I}_{\{\zeta > t\}} v(t, X_t; T), \qquad v(t, X_t; T) = \mathbb{E}[e^{-\int_t^T \gamma(s, X_s) ds} | X_t].$$

We add the parameter T to the arguments of v to indicate dependence of v on the maturity date T. Note that $v(t, x; T)$ is both the price of a bond and the *conditional survival probability*: $\mathbb{Q}(\zeta > T | X_t = x, \zeta > t)$. The *yield* $Y(t, x; T)$ of such a bond, on the set $\{\zeta > t\}$, is defined as

$$Y(t, x; T) := \frac{-\log v(t, x; T)}{T - t}. \tag{53}$$

The *credit spread* is defined as the yield minus the risk-free rate of interest. Obviously, in the case of zero interest rates, we have: yield = credit spread.

In Carr and Linetsky (2006), the authors introduce a class of unified credit-equity models known as *Jump to Default Constant Elasticity of Variance* or JDCEV. Specifically, in the time-homogeneous case, the underlying S is described by (1) with

$$a(x) = \frac{1}{2}\delta^2 e^{2\beta x}, \qquad \gamma(x) = b + c\,\delta^2 e^{2\beta x}, \qquad \nu(x, dz) = 0,$$

where $\delta > 0$, $b \geq 0$, $c \geq 0$. We will restrict our attention to cases in which $\beta < 0$. From a financial perspective, this restriction makes sense, as it results in volatility and default intensity *increasing* as $S \to 0^+$, which is consistent with the leverage effect. Note that when $c > 0$, the asset S may only go to zero via a jump from a strictly positive value. That is, according to the Feller boundary classification for one-dimensional diffusions (see Borodin and Salminen (2002), p.14), the endpoint $-\infty$ is a *natural boundary* for the killed diffusion X (i.e., the probability that X reaches $-\infty$ in finite time is zero). The survival probability $v(t, x; T)$ in this setting is computed in Mendoza-Arriaga et al. (2010), equation (8.13). We have

$$\begin{aligned} v(t, x; T) &= u(T - t, x) \\ &= \sum_{n=0}^{\infty} \left(e^{-(b+\omega n)(T-t)} \frac{\Gamma(1 + c/|\beta|)\Gamma(n + 1/(2|\beta|))}{\Gamma(\nu + 1)\Gamma(1/(2|\beta|))n!} \right. \\ &\qquad \left. \times A^{1/(2|\beta|)} e^x \exp\left(-Ae^{-2\beta x}\right) {}_1F_1(1 - n + c/|\beta|; \nu + 1; Ae^{-2\beta x}) \right) \end{aligned} \tag{54}$$

where ${}_1F_1$ is the Kummer confluent hypergeometric function, $\Gamma(x)$ is a Gamma function and

$$\nu = \frac{1 + 2c}{2|\beta|}, \qquad A = \frac{b}{\delta^2|\beta|}, \qquad \omega = 2|\beta|b.$$

We compute $u(T-t,x)$ using both equation (54) (truncating the infinite series at $n=70$) as well as using Corollary 10. We use basis functions from the Taylor series expansion of Example 5: $B_n(x) = (x-\bar{x})^n$. After computing bond prices, we then calculate the corresponding credit spreads using (53). Approximate spreads are denoted

$$Y^{(n)}(t,x;T) := \frac{-\log v^{(n)}(t,x;T)}{T-t}.$$

The survival probabilities are and the corresponding yields are plotted in Figure 6. Values for the yields from Figure 6 can also be found in Table 3.

Remark 23. To compute survival probabilities $v(t,x;T)$, one assumes a payoff function $h(x)=1$. Note that the Fourier transform of a constant is simply a Dirac delta function: $\widehat{h}(\xi) = \delta(\xi)$. Thus, when computing survival probabilities and/or credit spreads, no numerical integration is required. Rather, one simply uses the following identity

$$\int_{\mathbb{R}} \widehat{u}(\xi)\partial_\xi^n \delta(\xi) d\xi = (-1)^n \partial_\xi^n \widehat{u}(\xi)|_{\xi=0}.$$

to compute inverse Fourier transforms. From the above identity and equations (32) - (33) one easily obtains

$$\begin{aligned}
u_0(t,x) &= e^{-\left(b+\delta^2 c e^{2x\beta}\right)t},\\
u_1(t,x) &= e^{-\left(b+\delta^2 c e^{2x\beta}\right)t}\Big(-\delta^2 b c e^{2x\beta} t^2 \beta + \frac{1}{2}\delta^4 c e^{4x\beta} t^2 \beta - \delta^4 c^2 e^{4x\beta} t^2 \beta\Big),\\
u_2(t,x) &= e^{-\left(b+\delta^2 c e^{2x\beta}\right)t}\Big(-\delta^4 c e^{4x\beta} t^2 \beta^2 - \frac{2}{3}\delta^2 b^2 c e^{2x\beta} t^3 \beta^2 + \delta^4 b c e^{4x\beta} t^3 \beta^2 - 2\delta^4 b c^2 e^{4x\beta} t^3 \beta^2\\
&\quad - \frac{1}{3}\delta^6 c e^{6x\beta} t^3 \beta^2 + 2\delta^6 c^2 e^{6x\beta} t^3 \beta^2 - \frac{4}{3}\delta^6 c^3 e^{6x\beta} t^3 \beta^2 + \frac{1}{2}\delta^4 b^2 c^2 e^{4x\beta} t^4 \beta^2\\
&\quad - \frac{1}{2}\delta^6 b c^2 e^{6x\beta} t^4 \beta^2 + \delta^6 b c^3 e^{6x\beta} t^4 \beta^2 + \frac{1}{8}\delta^8 c^2 e^{8x\beta} t^4 \beta^2 - \frac{1}{2}\delta^8 c^3 e^{8x\beta} t^4 \beta^2 + \frac{1}{2}\delta^8 c^4 e^{8x\beta} t^4 \beta^2\Big).
\end{aligned}$$

7.5 Implied Volatility Expansion for CEV

In this Section we apply the implied volatility expansion of Section 6 to the CEV model of Cox (1975). The log-price dynamics are given by

$$a(x) = \frac{1}{2}\delta^2 e^{2(\beta-1)x}, \qquad \nu(x,dz) = 0, \qquad \gamma(x) = 0, \qquad \delta > 0, \qquad \beta \in [0,1],$$

In this setting the exact price of a call option with strike $K = e^k$ can be expressed as follows:

$$\begin{aligned}
u(t,x;K) &= e^x Q(\kappa, 2 + \tfrac{2}{2-\beta}, 2\chi) - e^k\left(1 - Q(2\chi, \tfrac{2}{2-\beta}, 2\kappa)\right), \qquad (55)\\
Q(w,v,\mu) &= \sum_{n=0}^{\infty}\left(\frac{(\mu/2)^n e^{-\mu/2}}{n!}\frac{\Gamma(v/2+n, w/2)}{\Gamma(v/2+n)}\right),\\
\chi &= \frac{2e^{(2-\beta)x}}{\delta^2(2-\beta)^2 t},\\
\kappa &= \frac{2e^{(2-\beta)k}}{\delta^2(2-\beta)^2 t},
\end{aligned}$$

where $\Gamma(a)$ and $\Gamma(a,b)$ are the complete and incomplete Gamma functions respectively. For a given call option, the (almost) exact implied volatility σ corresponding to the CEV model can be obtained by solving (48) numerically, with $u(t,x;K)$ given by (55).

Hagan and Woodward (1999) derive the following implied volatility expansion for the CEV model

$$\sigma^{HW} = \frac{\delta}{f^{1-\beta}}\left(1 + \frac{(1-\beta)(2+\beta)}{24}\left(\frac{e^x - e^k}{f}\right)^2 + \frac{(1-\beta)^2}{24}\frac{\delta^2 t}{f^{2(1-\beta)}} + \cdots\right), \qquad f = \frac{1}{2}(e^x + e^k). \quad (56)$$

Our second order implied volatility expansion can be computed using equation (44). We compare the 2nd order implied volatility expansion to the Hagan-Woodward expansion (56) in Figures 7 and 8. Over the strikes and maturities tested, our implied volatility expansion performs favorably.

8 Conclusion

In this paper, we consider an asset whose risk-neutral dynamics are described by an exponential Lévy-type martingale subject to default. This class includes nearly all non-negative Markov processes. In this very general setting, we provide a family of approximations – one for each choice of the basis functions (i.e. Taylor, two-point Taylor, L^2 basis, etc.) – for (i) the transition density of the underlying (ii) European-style option prices and their sensitivities and (iii) defaultable bond prices and their credit spreads. For the transition densities, and thus for option and bond prices as well, we establish the accuracy of our asymptotic expansion. We also derive, for local volatility models, an expansion for the implied volatility of European calls/puts. Finally, we provide extensive numerical examples illustrating both the versatility and effectiveness of our methods.

A Proof of Theorem 8

Proof. The formal adjoint of $\mathcal{A}_n$ is defined as the operator $\mathcal{A}_n^\dagger$ such that

$$\langle f, \mathcal{A}_n g\rangle = \langle \mathcal{A}_n^\dagger f, g\rangle, \qquad \langle u, v\rangle = \int_{\mathbb{R}} \overline{u(x)}\, v(x)dx, \qquad u, v \in \mathcal{S}(\mathbb{R}).$$

The operator $\mathcal{A}_n^\dagger$ can be obtained formally by integrating by parts, which results in making the change $\partial_x \to -\partial_x$. That is

$$\mathcal{A}_n^\dagger = \phi_n(t, -\mathcal{D}), \qquad \mathcal{D} = -i\partial_x,$$

with ϕ_n as in (12). Note also that by (15) and (17) we have

$$\mathcal{A}_n^\dagger e^{-ix\xi} = \phi_n(t,\xi)e^{-ix\xi}, \qquad B_n(x)e^{-ix\xi} = B_n(i\partial_\xi)e^{-ix\xi}. \quad (57)$$

In the particular case of the expansion in Example 5, we have explicitly

$$\mathcal{A}_n^\dagger = \gamma_n(t)(-\partial_x - 1) + a_n(t)(\partial_x^2 + \partial_x) + \int_{\mathbb{R}} \nu_n(t,dz)(e^z - 1 - z)\partial_x + \int_{\mathbb{R}} \nu_n(t,dz)(e^{-z\partial_x} - 1 + z\partial_x).$$

We Fourier transform equation (27). First, we focus on the left-hand side

$$\begin{aligned}
\int_{\mathbb{R}} dx \tfrac{1}{\sqrt{2\pi}} e^{-i\xi x} \left(\partial_t + \mathcal{A}_0\right) v_n(t,x) &= \partial_t \int_{\mathbb{R}} dx \tfrac{1}{\sqrt{2\pi}} e^{-i\xi x} v_n(t,x) + \int_{\mathbb{R}} dx \left(\mathcal{A}_0^\dagger \tfrac{1}{\sqrt{2\pi}} e^{-i\xi x}\right) v_n(t,x) \\
&= \partial_t \widehat{v}(t,\xi) + \phi_0(t,\xi) \int_{\mathbb{R}} dx \tfrac{1}{\sqrt{2\pi}} e^{-i\xi x} v_n(t,x) \quad \text{(by (57))} \\
&= \left(\partial_t + \phi_0(t,\xi)\right) \widehat{v}_n(t,\xi).
\end{aligned}$$

Next, for the right-hand side of (27), by (57) we have

$$\begin{aligned}
-\sum_{k=1}^{n} \int_{\mathbb{R}} dx \left(\frac{e^{-i\xi x}}{\sqrt{2\pi}} B_k(x)\right) \mathcal{A}_k v_{n-k}(t,x) &= -\sum_{k=1}^{n} \int_{\mathbb{R}} dx \left(B_k(i\partial_\xi) \frac{e^{-i\xi x}}{\sqrt{2\pi}}\right) \mathcal{A}_k v_{n-k}(t,x) \\
&= -\sum_{k=1}^{n} B_k(i\partial_\xi) \int_{\mathbb{R}} dx \left(\mathcal{A}_k^\dagger \frac{e^{-i\xi x}}{\sqrt{2\pi}}\right) v_{n-k}(t,x) \\
&= -\sum_{k=1}^{n} B_k(i\partial_\xi) \left(\phi_k(t,\xi) \int_{\mathbb{R}} dx \frac{e^{-i\xi x}}{\sqrt{2\pi}} v_{n-k}(t,x)\right) \quad \text{(by (57))} \\
&= -\sum_{k=1}^{n} B_k(i\partial_\xi) \left(\phi_k(t,\xi) \widehat{v}_{n-k}(t,\xi)\right)
\end{aligned}$$

Thus, we have the following ODEs (in t) for $\widehat{v}_n(t,\xi)$

$$\left(\partial_t + \phi_0(t,\xi)\right) \widehat{v}_0(t,\xi) = 0, \qquad \widehat{v}_0(T,\xi) = \widehat{h}(\xi), \tag{58}$$

$$\left(\partial_t + \phi_0(t,\xi)\right) \widehat{v}_n(t,\xi) = -\sum_{k=1}^{n} B_k(i\partial_\xi) \left(\phi_k(t,\xi) \widehat{v}_{n-k}(t,\xi)\right) \qquad \widehat{v}_n(T,\xi) = 0. \tag{59}$$

One can easily verify (e.g., by substitution) that the solutions of (58) and (59) are given by (28) and (29) respectively.

□

B Mathematica Code

The following Mathematica code can be used to generate the $\widehat{u}_n(t,\xi)$ automatically:

$$\begin{aligned}
&\hat{u}[\mathrm{n_}, \mathrm{t_}, \lambda_] := \mathrm{Exp}\left[\mathrm{t}\phi_0[\lambda]\right] \sum_{\mathrm{k}=1}^{\mathrm{n}} \\
&\qquad \int_0^{\mathrm{t}} \mathrm{Exp}\left[-\mathrm{s}\phi_0[\lambda]\right] \left(\sum_{\mathrm{m}=0}^{\mathrm{k}} \frac{\mathrm{k}!}{\mathrm{m}!(\mathrm{k}-\mathrm{m})!} (-\bar{x})^{\mathrm{k}-\mathrm{m}} i^{\mathrm{m}} \mathrm{D}\left[\phi_{\mathrm{k}}[\lambda] u[\mathrm{n}-\mathrm{k}, \mathrm{s}, \lambda], \{\lambda, \mathrm{m}\}\right]\right) d\mathrm{s}; \\
&\hat{u}[0, \mathrm{t_}, \lambda_] = \mathrm{Exp}\left[\mathrm{t}\phi_0[\lambda]\right] \mathrm{h}[\lambda];
\end{aligned}$$

The function $\widehat{u}(t,n,\lambda)$ is now computed explicitly by typing $\hat{u}[\mathrm{n},\mathrm{t},\lambda]$ and pressing Shift+Enter.

C Proof of Theorem 14

Proof. For sake of simplicity we only prove the assertion when the default intensity and mean jump size are zero $\gamma = \mu = 0$, when the jump intensity and diffusion component are time-independent $a(t,x) \equiv a(x)$,

$\lambda(t,x) \equiv \lambda(x)$ and when the standard deviation of the jumps is constant $\delta(t,x) \equiv \delta$. Thus we consider the integro-differential operator

$$Lu(t,x) = \partial_t u(t,x) + \frac{a(x)}{2}(\partial_{xx} - \partial_x)u(t,x) - \lambda(x)\left(e^{\frac{\delta^2}{2}} - 1\right)\partial_x u(t,x) \\ + \lambda(x)\int_{\mathbb{R}} \big(u(t,x+z) - u(t,x)\big)\nu_{\delta^2}(dz),$$

with

$$\nu_{\delta^2}(dz) = \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{z^2}{2\delta^2}} dz.$$

Our idea is to use our expansion as a *parametrix*. That is, our expansion will as the starting point of the classical iterative method introduced by Levi (1907) to construct the fundamental solution $p(t,x;T,y)$ of L. Specifically, as in Pagliarani et al. (2013), we take as a parametrix our N-th order approximation $p^{(N)}(t,x;T,y)$ with basis functions $B_n = (x - \bar{x})^n$ and with $\bar{x} = y$. We first prove the case $N = 1$. By analogy with the classical approach (see, for instance, Friedman (1964) and Di Francesco and Pascucci (2005), Pascucci (2011) for the pure diffusive case, or Garroni and Menaldi (1992) for the integro-differential case), we have

$$p(t,x;T,y) = p^{(1)}(t,x;T,y) + \int_t^T \int_{\mathbb{R}} p^{(0)}(t,x;s,\xi)\Phi(s,\xi;T,y)d\xi ds, \tag{60}$$

where Φ is determined by imposing the condition

$$0 = Lp(t,x;T,y) = Lp^{(1)}(t,x;T,y) + \int_t^T \int_{\mathbb{R}} Lp^{(0)}(t,x;s,\xi)\Phi(s,\xi;T,y)d\xi ds - \Phi(t,x;T,y).$$

Equivalently, we have

$$\Phi(t,x;T,y) = Lp^{(1)}(t,x;T,y) + \int_t^T \int_{\mathbb{R}} Lp^{(0)}(t,x;s,\xi)\Phi(s,\xi;T,y)d\xi ds,$$

and therefore by iteration

$$\Phi(t,x;T,y) = \sum_{n=0}^{\infty} Z_n(t,x;T,y), \tag{61}$$

where

$$Z_0(t,x;T,y) := Lp^{(1)}(t,x;T,y), \tag{62}$$

$$Z_{n+1}(t,x;T,y) := \int_t^T \int_{\mathbb{R}} Lp^{(0)}(t,x;s,\xi)Z_n(s,\xi;T,y)d\xi ds. \tag{63}$$

The proof of Theorem 14 is based on several technical lemmas which provide pointwise bounds of each term Z_n in (61). These bounds combined with formula (60) give the estimate of $\left|p(t,x;T,y) - p^{(1)}(t,x;T,y)\right|$.

For any $\alpha, \theta > 0$ and $\ell \geq 0$, consider the integro-differential operators

$$L^{\alpha,\theta,\ell}u(t,x) = \partial_t u(t,x) + \frac{\alpha}{2}(\partial_{xx} - \partial_x)u(t,x) - \ell\left(e^{\frac{\theta}{2}} - 1\right)\partial_x u(t,x) + \ell\int_{\mathbb{R}} \big(u(t,x+z) - u(t,x)\big)\nu_\theta(dz),$$

$$\overline{L}^{\alpha,\theta,\ell}u(t,x) = \partial_t u(t,x) + \frac{\alpha}{2}\partial_{xx}u(t,x) + \ell\int_{\mathbb{R}} \big(u(t,x+z) - u(t,x)\big)\nu_\theta(dz).$$

The function $\Gamma^{\alpha,\theta,\ell}(t,x;T,y) := \Gamma^{\alpha,\theta,\ell}(T-t,x-y)$ where

$$\Gamma^{\alpha,\theta,\ell}(t,x) := e^{-\ell t}\sum_{n=0}^{\infty}\frac{(\ell t)^n}{n!}\Gamma_n^{\alpha,\theta,\ell}(t,x),$$

$$\Gamma_n^{\alpha,\theta,\ell}(t,x) := \frac{1}{\sqrt{2\pi(\alpha t+n\theta)}}\exp\left(-\frac{\left(x-\left(\frac{\alpha}{2}+\ell e^{\frac{\theta}{2}}-\ell\right)t\right)^2}{2(\alpha t+n\theta)}\right),$$

is the fundamental solution of $L^{\alpha,\theta,\ell}$. Analogously, the function $\bar{\Gamma}^{\alpha,\theta,\ell}(t,x;T,y) := \bar{\Gamma}^{\alpha,\theta,\ell}(T-t,x-y)$ where

$$\bar{\Gamma}^{\alpha,\theta,\ell}(t,x) := e^{-\ell t}\sum_{n=0}^{\infty}\frac{(\ell t)^n}{n!}\bar{\Gamma}_n^{\alpha,\theta}(t,x),$$

$$\bar{\Gamma}_n^{\alpha,\theta}(t,x) := \frac{1}{\sqrt{2\pi(\alpha t+n\theta)}}\exp\left(-\frac{x^2}{2(\alpha t+n\theta)}\right),$$

is the fundamental solution of $\overline{L}^{\alpha,\theta,\ell}$. Note that under our assumptions, at order zero we have

$$p^{(0)}(t,x;T,y) = \Gamma^{a(y),\delta^2,\lambda(y)}(t,x;T,y). \tag{64}$$

We also introduce the convolution operator $\mathcal{C}_\theta$ defined as

$$\mathcal{C}_\theta f(x) = \int_{\mathbb{R}} f(x+z)\frac{1}{\sqrt{2\pi\theta}}e^{-\frac{z^2}{2\theta}}dz. \tag{65}$$

Note that, for any $\theta > 0$, we have

$$\mathcal{C}_\theta\Gamma^{\alpha,\theta,\ell}(t,\cdot)(x) = e^{-\ell t}\sum_{n=0}^{\infty}\frac{(\ell t)^n}{n!}\Gamma_{n+1}^{\alpha,\theta,\ell}(t,x),$$

$$\mathcal{C}_\theta\bar{\Gamma}^{\alpha,\theta,\ell}(t,\cdot)(x) = e^{-\ell t}\sum_{n=0}^{\infty}\frac{(\ell t)^n}{n!}\bar{\Gamma}_{n+1}^{\alpha,\theta}(t,x).$$

In the rest of the section, we will always assume that

$$m \le \alpha,\theta \le M, \qquad 0 \le \ell \le M. \tag{66}$$

Even if not explicitly stated, all the constants appearing in the estimates (67), (68), (69), (72), (73) and (77) of the following lemmas will depend also on m and M.

Lemma 24. *For any $T > 0$ and $c > 1$ there exists a positive constant C such that*[6]

$$\mathcal{C}_\theta^N\Gamma^{\alpha,\theta,\ell}(t,x) \le C\,\mathcal{C}_{cM}^N\bar{\Gamma}^{cM,cM,cM}(t,x), \tag{67}$$

for any $t \in (0,T]$, $x \in \mathbb{R}$ and $N \ge 0$.

Proof. For any $n \ge 0$ we have

$$\Gamma_n^{\alpha,\theta,\ell}(t,x) \le \sqrt{\frac{cM}{m}}q_n(t,x)\bar{\Gamma}_n^{cM,cM}(t,x),$$

[6] Here $\mathcal{C}_\theta^0$ denotes the identity operator.

where

$$q_n(t,x) = \exp\left(-\frac{\left(x - \left(\frac{\alpha}{2} + \ell e^{\frac{\theta}{2}} - \ell\right) t\right)^2}{2(\alpha t + n\theta)} + \frac{x^2}{2cM(t+n)} \right).$$

A direct computation shows that

$$\max_{x\in\mathbb{R}} q_n(t,x) = \exp\left(\frac{s^2 \left(\alpha + 2\left(e^{\frac{\theta}{2}} - 1\right)\ell\right)^2}{8\left(cM(n+s) - s\alpha - n\delta^2\right)} \right) \leq \exp\left(\frac{T\left(\alpha + 2\left(e^{\frac{\theta}{2}} - 1\right)\ell\right)^2}{8(cM-\alpha)} \right),$$

for any $t \in (0,T]$, $n \geq 0$ and α, θ, ℓ in (66). Then the thesis is a straightforward consequence of the fact that $q_n(t,x)$ is bounded on $(0,T] \times \mathbb{R}$, uniformly with respect to $n \geq 0$ and α, θ, ℓ in (66). $\square$

Lemma 25. *For any $T > 0$, $k \in \mathbb{N}$ and $c > 1$, there exists a positive constant C such that*

$$\left|\partial_x^k \Gamma_n^{\alpha,\theta,\ell}(t,x)\right| \leq \frac{C}{(\alpha t + n\theta)^{k/2}} \Gamma_n^{c\alpha,c\theta,\ell}(t,x), \tag{68}$$

for any $x \in \mathbb{R}$, $t \in]0,T]$ and $n \geq 0$.

Proof. For any $k \geq 1$ we have

$$\partial_x^k \Gamma_n^{\alpha,\theta,\ell}(t,x) = \frac{1}{(\alpha t + n\theta)^{k/2}} \Gamma_n^{\alpha,\theta,\ell}(t,x) p_k \left(\frac{x - \left(\frac{\alpha}{2} + \ell e^{\frac{\theta}{2}} - \ell\right) t}{\sqrt{\alpha t + n\theta}} \right),$$

where p_k is a polynomial of degree k. To prove the Lemma we will show that there exists a positive constant C, which depends only on m, M, T, c and k, such that

$$\left(\frac{\left|x - \left(\frac{\alpha}{2} + \ell e^{\frac{\theta}{2}} - \ell\right) t\right|}{\sqrt{\alpha t + n\theta}} \right)^j \Gamma_n^{\alpha,\theta,\ell}(t,x) \leq C\, \Gamma_n^{c\alpha,c\theta,\ell}(t,x), \qquad j \leq k.$$

Proceeding as above, we set

$$\left(\frac{\left|x - \left(\frac{\alpha}{2} + \ell e^{\frac{\theta}{2}} - \ell\right) t\right|}{\sqrt{\alpha t + n\theta}} \right)^j \Gamma_n^{\alpha,\theta,\ell}(t,x) = \Gamma_n^{c\alpha,c\theta,\ell}(t,x) q_{n,j}(t,x),$$

where

$$q_{n,j}(t,x) = \left(\frac{\left|x - \left(\frac{\alpha}{2} + \ell e^{\frac{\theta}{2}} - \ell\right) t\right|}{\sqrt{\alpha t + n\theta}} \right)^j \exp\left(-\frac{\left(x - \left(\frac{\alpha}{2} + \ell e^{\frac{\theta}{2}} - \ell\right) t\right)^2}{2(\alpha t + n\theta)} + \frac{\left(x - \left(\frac{c\alpha}{2} + \ell e^{\frac{c\theta}{2}} - \ell\right) t\right)^2}{2(c\alpha t + nc\theta)} \right).$$

Then the thesis follows from the boundedness of $q_{n,j}$ on $(0,T] \times \mathbb{R}$, uniformly with respect to $n \geq 0$ and α, θ, ℓ in (66). Indeed the maximum of $q_{n,j}$ can be computed explicitly and we have

$$\lim_{n\to\infty} \left(\max_{x\in\mathbb{R},\ t\in]0,T]} q_{n,j}(t,x) \right) = \left(\frac{cj}{(c-1)e} \right)^{\frac{j}{2}}.$$

$\square$

Lemma 26. *For any $T>0$ and $N\in\mathbb{N}$, there exists a positive constant C such that*

$$\ell t\, \mathcal{C}_\theta^N \Gamma^{\alpha,\theta,\ell}(t,x) \le C\,\Gamma^{\alpha,2(N+1)\theta,\ell}(t,x) \tag{69}$$

for any $t\in(0,T]$ and $x\in\mathbb{R}$.

Proof. We first prove there exists a constant C_0, which depends only on m,M,T and N, such that

$$\Gamma^{\alpha,\theta,\ell}_{n+N}(t,x) \le C_0\,\Gamma^{\alpha,2(N+1)\theta,\ell}_n(t,x), \tag{70}$$

$$\Gamma^{\alpha,\theta,\ell}_{N}(t,x) \le C_0\,\Gamma^{\alpha,2(N+1)\theta,\ell}_1(t,x), \tag{71}$$

for any $t\in]0,T]$, $x\in\mathbb{R}$, $n\ge 1$ and α,θ,ℓ in (66). To prove (70) we observe that

$$\Gamma^{\alpha,\theta,\ell}_{n+N}(t,x) \le \frac{1}{\sqrt{2\pi(\alpha t+(n+N)\theta)}}\exp\left(-\frac{\left(x-\left(\frac{\alpha}{2}+\ell e^{(N+1)\theta}-\ell\right)t\right)^2}{2(\alpha t+2n(N+1)\theta)}\right)q_n(t,x),$$

where

$$q_n(t,x)=\exp\left(-\frac{\left(x-\left(\frac{\alpha}{2}+\ell e^{\frac{\theta}{2}}-\ell\right)t\right)^2}{2(\alpha t+(n+N)\theta)}+\frac{\left(x-\left(\frac{\alpha}{2}+\ell e^{(N+1)\theta}-\ell\right)t\right)^2}{2(\alpha t+2n(N+1)\theta)}\right).$$

Now it is easy to check that

$$\max_{x\in\mathbb{R}} q_n(t,x)=\exp\left(\frac{\left(e^{(1+N)\theta}-e^{\frac{\theta}{2}}\right)^2 t^2\ell^2}{2(n-N+2nN)\theta}\right)\le \exp\left(\frac{\left(e^{(1+N)\theta}-e^{\frac{\theta}{2}}\right)^2 t^2\ell^2}{2N\theta}\right).$$

for any $t\ge 0$. Thus q_n is bounded on $(0,T]\times\mathbb{R}$, uniformly with respect to $n\in\mathbb{N}$ and α,θ,ℓ in (66). To see the above bound, simply observe that

$$\frac{\sqrt{\alpha t+2n(N+1)\theta}}{\sqrt{\alpha t+(N+n)\theta}}\le\sqrt{2(N+1)}.$$

The proof of (71) is completely analogous. Finally, by (70)-(71) we have

$$\begin{aligned}
\ell t\, \mathcal{C}_\theta^N \Gamma^{\alpha,\theta,\ell}(t,x) &= e^{-\ell t}\ell t\,\Gamma^{\alpha,\theta,\ell}_N(t,x)+\ell t\, e^{-\ell t}\sum_{n=1}^{\infty}\frac{(\ell t)^n}{n!}\Gamma^{\alpha,\theta,\ell}_{n+N}(t,x)\\
&\le C_0\left(e^{-\ell t}\ell t\,\Gamma^{\alpha,2(N+1)\theta,\ell}_1(t,x)+\ell t\, e^{-\ell t}\sum_{n=1}^{\infty}\frac{(\ell t)^n}{n!}\Gamma^{\alpha,2(N+1)\theta,\ell}_{n}(t,x)\right)\\
&\le C_0(1+MT)\,\Gamma^{\alpha,2(N+1)\theta,\ell}(t,x).
\end{aligned}$$

$\square$

Lemma 27. *For any $T>0$ and $N\ge 2$, there exists a positive constant C such that*

$$\mathcal{C}_\theta^N \Gamma^{\alpha,\theta,\ell}(t,x) \le C\,\mathcal{C}_{2N\theta}\Gamma^{\alpha,2N\theta,\ell}(t,x) \tag{72}$$

for any $t\in(0,T]$ and $x\in\mathbb{R}$.

Proof. By (70)

$$\mathcal{C}_\theta^N\Gamma^{\alpha,\theta,\ell}(t,x) = e^{-\ell t}\sum_{n=0}^{\infty}\frac{(\ell t)^n}{n!}\Gamma^{\alpha,\theta,\ell}_{n+1+(N-1)}(t,x) \le C\, e^{-\ell t}\sum_{n=0}^{\infty}\frac{(\ell t)^n}{n!}\Gamma^{\alpha,2N\theta,\ell}_{n+1}(t,x) = C\,\mathcal{C}_{2N\theta}\Gamma^{\alpha,2N\theta,\ell}(t,x).$$

□

Lemma 28. *For any $T>0$, $N\ge 1$ and $c>1$, there exists a positive constant C such that*

$$\left(\frac{|x|}{\sqrt{\alpha t+n\theta}}\right)^N\Gamma_n^{\alpha,\theta,\ell}(t,x)\le C\Gamma_n^{c\alpha,c\theta,\ell}(t,x), \tag{73}$$

for any $x\in\mathbb{R}$, $t\in(0,T]$ and $n\ge 0$.

Proof. We first show that there exist three constants $C_1=C_1(m,M,T,N,c)$, $C_2=C_2(N,c)$ and $C_3=C_3(m,M,T,N,c)$ such that

$$e^{-\frac{\left(x-\left(\frac{\alpha}{2}+\ell e^{\frac{\theta}{2}}-\ell\right)t\right)^2}{2(\alpha t+n\theta)}}\le C_1 e^{-\frac{x^2}{2c^{1/3}(\alpha T+n\theta)}}, \tag{74}$$

$$\left(\frac{|x|}{\sqrt{\alpha t+n\theta}}\right)^N e^{-\frac{x^2}{2c^{1/3}(\alpha T+n\theta)}}\le C_2 e^{-\frac{x^2}{2c^{2/3}(\alpha T+n\theta)}}, \tag{75}$$

$$e^{-\frac{x^2}{2c^{2/3}(\alpha T+n\theta)}}\le C_3 e^{-\frac{\left(x-\left(\frac{c\alpha}{2}+\ell e^{\frac{c\theta}{2}}-\ell\right)t\right)^2}{2c(\alpha t+n\theta)}}, \tag{76}$$

for any $x\in\mathbb{R}$, $t\in(0,T]$ and $n\ge 0$. In order to prove (74) we consider

$$q_n(t,x)=\exp\left(-\frac{\left(x-\left(\frac{\alpha}{2}+\ell e^{\frac{\theta}{2}}-\ell\right)t\right)^2}{2(\alpha t+n\theta)}+\frac{x^2}{2c^{1/3}(\alpha t+n\theta)}\right),$$

and show that

$$\max_{x\in\mathbb{R}}q_n(t,x)=\exp\left(\frac{\left(\frac{\alpha}{2}+\ell e^{\frac{\theta}{2}}-\ell\right)^2 t^2}{2(c^{1/3}-1)\,(t\alpha+n\theta)}\right)\le\exp\left(\frac{\left(\frac{\alpha}{2}+\ell e^{\frac{\theta}{2}}-\ell\right)^2 T}{2(c^{1/3}-1)}\right),$$

for any $t\in(0,T]$. Thus q_n is bounded on $(0,T]\times\mathbb{R}$, uniformly in $n\ge 0$ and α,θ,ℓ in (66). The proof of (76) is completely analogous. Equation (75) comes directly by setting

$$C_2=\max_{a\in\mathbb{R}^+}\left(a^N e^{-\frac{a^2}{2c^{1/3}}+\frac{a^2}{2c^{2/3}}}\right)=e^{-\frac{N}{2}}\left(\frac{c^{1/3}\sqrt{N}}{\sqrt{c^{1/3}-1}}\right)^N.$$

Now, by (74) we have

$$\left(\frac{|x|}{\sqrt{\alpha t+n\theta}}\right)^N\Gamma_n^{\alpha,\theta,\ell}(t,x)\le C_1\left(\frac{|x|}{\sqrt{\alpha t+n\theta}}\right)^N\frac{e^{-\frac{x^2}{2c^{1/3}(\alpha T+n\theta)}}}{\sqrt{2\pi(\alpha T+n\theta)}}$$

(by (75))

$$\le C_1C_2\frac{e^{-\frac{x^2}{2c^{2/3}(\alpha T+n\theta)}}}{\sqrt{2\pi(\alpha T+n\theta)}}$$

(by (76))

$$\leq C_1C_2C_3\sqrt{c}\,\Gamma_n^{c\alpha,c\theta,\ell}(t,x).$$

□

Lemma 29. *For any $T>0$, $c>1$ and $j\in\mathbb{N}\cup\{0\}$ there exists a positive constant C such that*

$$|x|\mathcal{C}_\theta^j\Gamma^{\alpha,\theta,\ell}(t,x)\leq C\left(\mathcal{C}_{2jc\theta}\Gamma^{c\alpha,c\theta,\ell}(t,x)+\mathcal{C}_{2(j+1)c\theta}\Gamma^{c\alpha,4c\theta,\ell}(t,x)\right),\tag{77}$$

for any $t\in(0,T]$ and $x\in\mathbb{R}$.

Proof. By Lemma 28 there is a constant C_0, only dependent on m,M,T and c, such that

$$\begin{aligned}|x|\mathcal{C}_\theta^j\Gamma^{\alpha,\theta,\ell}(t,x)&\leq C_0e^{-\ell t}\sum_{n=0}^{\infty}\frac{(\ell t)^n}{n!}\sqrt{\alpha t+(n+j)\theta}\Gamma_{n+j}^{c\alpha,c\theta,\ell}(t,x)\\&\leq C_0\sqrt{M}(\sqrt{T}+j)\mathcal{C}_{c\theta}^j\Gamma^{c\alpha,c\theta,\ell}(t,x)+C_0\sqrt{M}e^{-\ell t}\sum_{n=0}^{\infty}\frac{(\ell t)^n}{n!}n\Gamma_{n+j}^{c\alpha,c\theta,\ell}(t,x)\\&\leq C_0\sqrt{M}(\sqrt{T}+j)\mathcal{C}_{c\theta}^j\Gamma^{c\alpha,c\theta,\ell}(t,x)+C_0M^{\frac{3}{2}}T\mathcal{C}_{c\theta}^{j+1}\Gamma^{c\alpha,c\theta,\ell}(t,x),\end{aligned}$$

for any $t\in(0,T]$ and $x\in\mathbb{R}$ and α,θ,ℓ in (66). Therefore, the thesis follows from Lemma 26 for $j=0$ and from Lemma 27 for $j\geq 1$. □

Lemma 30. *For any $T>0$ and $N,k\geq 1$ we have*

$$\mathcal{C}_\theta^N\bar{\Gamma}^{\alpha,\theta,\ell}(t,x)\leq\sqrt{k+1}\,\mathcal{C}_\theta^{N+k}\bar{\Gamma}^{\alpha,\theta,\ell}(t,x),\qquad t\in]0,T],\ x\in\mathbb{R}.$$

Proof. A direct computation shows that

$$\max_{x\in\mathbb{R}}\frac{\bar{\Gamma}_{n+N}^{\alpha,\theta}(t,x)}{\bar{\Gamma}_{n+N+k}^{\alpha,\theta}(t,x)}=\frac{\sqrt{\alpha t+(n+N+k)\theta}}{\sqrt{\alpha t+(n+N)\theta}}\leq\sqrt{k+1},$$

for any $t\leq T$, $n\geq 0$, $N\geq 1$ and α,θ,ℓ in (66). This concludes the proof. □

Proposition 31. *The solution of (27) with $n=1$, $\mathcal{A}_1$ as in (14)-(13) and $h=\delta_y$, takes the form*

$$v_1(t,x;T,y)=\left((T-t)(x-y)+\frac{(T-t)^2}{2}J\right)\mathcal{A}_1p^{(0)}(t,x;T,y),$$

where J is the operator

$$J=a_0(2\partial_x-1)-\lambda_0\left(e^{\frac{\delta^2}{2}}-1+\delta^2\partial_x\mathcal{C}_{\delta^2}\right),\tag{78}$$

and $\mathcal{C}_{\delta^2}$ is the convolution operator defined in (65).

Proof. Under the assumptions in this appendix, ϕ_n in (13) takes the form

$$\phi_n(\xi)=a_n(-\xi^2-i\xi)-i\xi\lambda_n\left(e^{\frac{\delta^2}{2}}-1\right)+\lambda_n\left(e^{-\frac{\delta^2\xi^2}{2}}-1\right),\tag{79}$$

with

$$a_n = \frac{\partial^n a(y)}{n!}, \qquad \lambda_n = \frac{\partial^n \lambda(y)}{n!}.$$

Now, by (29) with $n = 1$, we have

$$\begin{aligned}
\widehat{v}_1(t, \xi; T, y) &= \int_t^T e^{(s-t)\phi_0(\xi)} \left(i\partial_\xi - y\right) \left(\phi_1(\xi)\widehat{p}^{(0)}(s, \xi; T, y)\right) ds \\
&= \left(i\partial_\xi - y\right) \int_t^T e^{(s-t)\phi_0(\xi)} \phi_1(\xi)\widehat{p}^{(0)}(s, \xi; T, y) ds \\
&\quad - i \int_t^T \left(\partial_\xi e^{(s-t)\phi_0(\xi)}\right) \phi_1(\xi)\widehat{p}^{(0)}(s, \xi; T, y) ds.
\end{aligned}$$

Then, recalling that

$$\widehat{p}^{(0)}(s, \xi; T, y) = \frac{1}{\sqrt{2\pi}} e^{(T-s)\phi_0(\xi) - i\xi y},$$

we get

$$\widehat{v}_1(t, \xi; T, y) = \left((T-t)\left(i\partial_\xi - y\right) - \frac{i(T-t)^2}{2}\phi_0'(\xi)\right) \phi_1(\xi)\widehat{p}^{(0)}(t, \xi; T, y).$$

The Proposition follows by (79) and by inverse Fourier transforming from ξ into the original coordinate x. □

Proposition 32. *For any $c > 1$ and $\tau > 0$, there exists a positive constant C, only dependent on $c, \tau, m, M, \|\lambda_1\|_\infty$ and $\|a_1\|_\infty$, such that*

$$\left|(x-y)^{2-n}(\partial_{xx} - \partial_x)v_n(t, x; T, y)\right| \le C(1 + \|\lambda_1\|_\infty \mathcal{C}_{cM})\, \bar{\Gamma}^{cM,cM,cM}(t, x; T, y), \tag{80}$$

for any $n \in \{0, 1\}$, $x, y \in \mathbb{R}$ and $t, T \in \mathbb{R}$ with $0 < T - t \le \tau$.

Proof. Recalling the expression of $v_0(t, x; T, y) \equiv p^{(0)}(t, x; T, y)$ in (64), the case $n = 0$ directly follows from Lemmas 25, 28 and 24 with $N = 0$. For the case $n = 1$, by Proposition 31 we have

$$\begin{aligned}
(x-y)(\partial_{xx} - \partial_x)v_1(t, x; T, y) &= (T-t)(x-y)\big((x-y)(\partial_{xx} - \partial_x) + 2\partial_x - 1\big)\mathcal{A}_1 p^{(0)}(t, x; T, y) \\
&\quad + \frac{(T-t)^2}{2}(x-y)J(\partial_{xx} - \partial_x)\mathcal{A}_1 p^{(0)}(t, x; T, y),
\end{aligned}$$

with J as in (78) and $\mathcal{A}_1$ acting as

$$\mathcal{A}_1 u(x) = a_1(\partial_{xx} - \partial_x)u(x) - \lambda_1 \left(\left(e^{\frac{\delta^2}{2}} - 1\right)\partial_x u(x) - \mathcal{C}_{\delta^2} u(x) + u(x)\right).$$

In the computations that follow below, in order to shorten notation, we omit the dependence of t, x, T, y in any function. By the commutative property of the operators ∂_x and $\mathcal{C}$, and by applying Lemmas 25, 26 and

28 with $N=1$, there exists a positive constant C_1 only dependent on $c,\tau,m,M,\|\lambda_1\|_\infty$ and $\|a_1\|_\infty$ such that

$$|(T-t)(x-y)\big((x-y)(\partial_{xx}-\partial_x)+2\partial_x-1\big)a_1(\partial_{xx}-\partial_x)p^{(0)}|\le C_1\Gamma^{ca(y),c\delta^2,\lambda(y)},$$

$$\left|(T-t)(x-y)\big((x-y)(\partial_{xx}-\partial_x)+2\partial_x-1\big)\lambda_1\left(\left(e^{\frac{\delta^2}{2}}-1\right)\partial_x+1\right)p^{(0)}\right|\le C_1\big(\Gamma^{ca(y),c\delta^2,\lambda(y)}+\Gamma^{ca(y),4c\delta^2,\lambda(y)}\big),$$

$$\frac{(T-t)^2}{2}|(x-y)J(\partial_{xx}-\partial_x)a_1(\partial_{xx}-\partial_x)p^{(0)}|\le C_1\big(\Gamma^{ca(y),c\delta^2,\lambda(y)}+\Gamma^{ca(y),4c\delta^2,\lambda(y)}\big),$$

$$\frac{(T-t)^2}{2}\left|(x-y)J(\partial_{xx}-\partial_x)\lambda_1\left(\left(e^{\frac{\delta^2}{2}}-1\right)\partial_x+1\right)p^{(0)}\right|\le C_1\big(\Gamma^{ca(y),c\delta^2,\lambda(y)}+\Gamma^{ca(y),4c\delta^2,\lambda(y)}\big), \tag{81}$$

for any $x,y\in\mathbb{R}$ and $t,T\in\mathbb{R}$ with $0<T-t\le\tau$. Analogously, by the commutative property of ∂_x and $\mathcal{C}$, and by applying Lemmas 28, 25, 29 and 27 with $N=2$, there exists a positive constant C_2 only dependent on $c,\tau,m,M,\|\lambda_1\|_\infty$ and $\|a_1\|_\infty$ such that

$$|(T-t)(x-y)\big((x-y)(\partial_{xx}-\partial_x)+2\partial_x-1\big)\lambda_1\mathcal{C}_{\delta^2}p^{(0)}|\|\lambda_1\|_\infty\le C_2\left(\mathcal{C}_{c\delta^2}\Gamma^{ca(y),c\delta^2,\lambda(y)}+\mathcal{C}_{4c\delta^2}\Gamma^{ca(y),4c\delta^2,\lambda(y)}\right),$$

$$\frac{(T-t)^2}{2}|(x-y)J(\partial_{xx}-\partial_x)\lambda_1\mathcal{C}_{\delta^2}p^{(0)}|\le\|\lambda_1\|_\infty C_2\big(\mathcal{C}_{c\delta^2}\Gamma^{ca(y),c\delta^2,\lambda(y)}+\mathcal{C}_{4c\delta^2}\Gamma^{ca(y),4c\delta^2,\lambda(y)}\big), \tag{82}$$

for any $x,y\in\mathbb{R}$ and $t,T\in\mathbb{R}$ with $0<T-t\le\tau$. Then, (80) follows from (81) and (82) by applying Lemma 24 with $N=0$ and $N=1$ respectively. □

Proposition 33. *For any $c>1$ and $\tau>0$, there exists a positive constant C, only dependent on $c,\tau,m,M,\|\lambda_1\|_\infty$ and $\|a_1\|_\infty$, such that*

$$\left|(x-y)^{2-n}\left(\left(e^{\frac{\delta^2}{2}}-1\right)\partial_x+\mathcal{C}_{\delta^2}-1\right)v_n(t,x;T,y)\right|\le C(1+\mathcal{C}_{cM})\,\bar{\Gamma}^{cM,cM,cM}(t,x;T,y), \tag{83}$$

for any $n\in\{0,1\}$, $x,y\in\mathbb{R}$ and $t,T\in\mathbb{R}$ with $0<T-t\le\tau$.

Proof. For simplicity we only prove the thesis for $n=0$. The proof for $n=1$ is entirely analogous to that of Proposition 32. Once again, hereafter we omit the dependence of t,x,T,y in any function we consider. Recalling the expression of $v_0(t,x;T,y)\equiv p^{(0)}(t,x;T,y)$ in (64), by Lemmas 25, 28 and 29, there exists a positive constant C_1 only dependent on c,τ,m,M such that

$$\left|(x-y)^2\left(\left(e^{\frac{\delta^2}{2}}-1\right)\partial_x+\mathcal{C}_{\delta^2}-1\right)v_0\right|\le C_1\big(\Gamma^{ca(y),4c\delta^2,\lambda(y)}+(1+\mathcal{C}_{16c\delta^2})\Gamma^{ca(y),16c\delta^2,\lambda(y)}\big),$$

for any $x,y\in\mathbb{R}$ and $t,T\in\mathbb{R}$ with $0<T-t\le\tau$. Then, (83) follows from Lemma 24 with $N=0$ and with $N=1$. □

Proposition 34. *For any $c>1$ and $\tau>0$, there exists a positive constant C, only dependent on $c,\tau,m,M,\|\lambda_1\|_\infty,\|\lambda_2\|_\infty$, $\|a_1\|_\infty$ and $\|a_2\|_\infty$, such that*

$$|Z_n(t,x;T,y)|\le\frac{C^{n+1}(T-t)^{\frac{n}{2}}}{\sqrt{n!}}\left(1+\|\lambda_1\|_\infty\mathcal{C}_{cM}^{n+1}\right)\bar{\Gamma}^{cM,cM,cM}(t,x;T,y), \tag{84}$$

for any $n \geq 0$, $x, y \in \mathbb{R}$ and $t, T \in \mathbb{R}$ with $0 < T - t \leq \tau$.

Proof. Let us define the operators

$$L_0 = \partial_t + \mathcal{A}_0, \qquad L_1 = \partial_t + \mathcal{A}_0 + (x-y)\mathcal{A}_1.$$

Let us recall that, by (26) and (27) with $n = 1$, we have

$$L_0 v_0 = 0, \qquad L_0 v_1 = -(L_1 - L_0)v_0.$$

Thus, by (62) we have

$$\begin{aligned} Z_0(t,x;T,y) = Lp^{(1)}(t,x;T,y) &= Lv_0(t,x;T,y) + Lv_1(t,x;T,y) \\ &= (L - L_1)v_0(t,x;T,y) + (L - L_0)v_1(t,x;T,y), \end{aligned}$$

where $(L - L_0)$ and $(L - L_1)$ are explicitly given by

$$\begin{aligned} (L - L_0) &= (a(x) - a(y))(\partial_{xx} - \partial_x) + (\lambda(x) - \lambda(y))\left(\left(e^{\frac{\delta^2}{2}} - 1\right)\partial_x + \mathcal{C}_{\delta^2} - 1\right), \qquad (85) \\ (L - L_1) &= (a(x) - a(y) - a'(y)(x-y))(\partial_{xx} - \partial_x) \\ &\quad + (\lambda(x) - \lambda(y) - \lambda'(y)(x-y))\left(\left(e^{\frac{\delta^2}{2}} - 1\right)\partial_x + \mathcal{C}_{\delta^2} - 1\right). \end{aligned}$$

Thus, by the Lipschitz assumptions on a, λ and their first order derivatives, we obtain

$$\begin{aligned} |Z_0(t,x;T,y)| \leq{}& \|a_2\|_\infty |x-y|^2 |(\partial_{xx} - \partial_x)v_0(t,x;T,y)| + \|a_1\|_\infty |x-y| |(\partial_{xx} - \partial_x)v_1(t,x;T,y)| \\ &+ \|\lambda_2\|_\infty |x-y|^2 \left|\left(\left(e^{\frac{\delta^2}{2}} - 1\right)\partial_x + \mathcal{C}_{\delta^2} - 1\right) v_0(t,x;T,y)\right| \\ &+ \|\lambda_1\|_\infty |x-y| \left|\left(\left(e^{\frac{\delta^2}{2}} - 1\right)\partial_x + \mathcal{C}_{\delta^2} - 1\right) v_1(t,x;T,y)\right|. \end{aligned}$$

and, as $\|\lambda_1\|_\infty = 0$ implies $\|\lambda_2\|_\infty = 0$, by Propositions 32 and 33 there exists a positive constant C, only dependent on c, τ, m, M,$\|\lambda_1\|_\infty$,$\|\lambda_2\|_\infty$, $\|a_1\|_\infty$ and $\|a_2\|_\infty$, such that (84) holds for $n = 0$. To prove the general case, we proceed by induction on n. First note that, by (26) we have

$$|Lp^{(0)}(t,x;T,y)| = |(L - L_0)p^{(0)}(t,x;T,y)|$$

(and by (85) and the Lipschitz property of α, λ)

$$\begin{aligned} &\leq \|a_1\|_\infty |x-y| |(\partial_{xx} - \partial_x)p^{(0)}(t,x;T,y)| \\ &\quad + \|\lambda_1\|_\infty |x-y| \left|\left(\left(e^{\frac{\delta^2}{2}} - 1\right)\partial_x + \mathcal{C}_{\delta^2} - 1\right) p^{(0)}(t,x;T,y)\right| \end{aligned}$$

(and by applying Lemmas 24, 25, 28 and 29 with $N = 0, 1$)

$$\leq C_0 \left(\frac{1}{\sqrt{T-t}} + \|\lambda_1\|_\infty \mathcal{C}_{cM}\right) \bar{\Gamma}^{cM,cM,cM}(t,x;T,y), \qquad (86)$$

for any $x, y \in \mathbb{R}$ and $t, T \in \mathbb{R}$ with $0 < T - t \leq \tau$, and where C_0 is a positive constant only dependent on $c, \tau, m, M, \|\lambda_1\|_\infty$ and $\|a_1\|_\infty$. Assume now (84) holds for $n \geq 0$. Then by (63) we obtain

$$|Z_{n+1}(t,x;T,y)| \leq \int_t^T \int_{\mathbb{R}} |Lp^{(0)}(t,x;s,\xi)||Z_n(s,\xi;T,y)| d\xi ds$$

(and by inductive hypothesis and by (86))

$$\leq \frac{C^{n+1}C_0}{\sqrt{n!}} \int_t^T (T-s)^{\frac{n}{2}} \int_{\mathbb{R}} \left(\frac{1}{\sqrt{s-t}} + \|\lambda_1\|_\infty \mathcal{C}_{cM} \right) \bar{\Gamma}^{cM,cM,cM}(t,x;s,\xi) \cdot \\ \cdot \left(1 + \|\lambda_1\|_\infty \mathcal{C}_{cM}^{n+1}\right) \bar{\Gamma}^{cM,cM,cM}(s,\xi;T,y) d\xi ds.$$

Now, by the semigroup property

$$\int_{\mathbb{R}} \mathcal{C}_\theta^k \bar{\Gamma}^{\alpha,\theta,\ell}(t,x;s,\xi) \mathcal{C}_\theta^N \bar{\Gamma}^{\alpha,\theta,\ell}(s,\xi;T,y)\, d\xi = \mathcal{C}_\theta^{k+N} \bar{\Gamma}^{\alpha,\theta,\ell}(t,x;T,y), \qquad k, N \geq 0, \tag{87}$$

and by the fact that[7]

$$\int_t^T \frac{(T-s)^{\frac{n}{2}}}{\sqrt{s-t}} ds = \frac{\sqrt{\pi}(T-t)^{\frac{n+1}{2}} \Gamma_E\left(\frac{2+n}{2}\right)}{\Gamma_E\left(\frac{3+n}{2}\right)} \leq \frac{\kappa (T-t)^{\frac{n+1}{2}}}{\sqrt{n+1}},$$

with $\kappa = \sqrt{2}\pi$, we obtain

$$\begin{aligned} |Z_{n+1}(t,x;T,y)| \leq{}& \frac{C^{n+1}C_0}{\sqrt{n!}} \left(\frac{\kappa (T-t)^{\frac{n+1}{2}}}{\sqrt{n+1}} \left(1 + \|\lambda_1\|_\infty \mathcal{C}_{cM}^{n+1}\right) \right) \bar{\Gamma}^{cM,cM,cM}(t,x;T,y) \\ &+ \frac{C^{n+1}C_0}{\sqrt{n!}} \left(\frac{2(T-t)^{\frac{n+2}{2}}}{n+2} \|\lambda_1\|_\infty \left(\mathcal{C}_{cM} + \|\lambda_1\|_\infty \mathcal{C}_{cM}^{n+2}\right) \right) \bar{\Gamma}^{cM,cM,cM}(t,x;T,y). \end{aligned} \tag{88}$$

Now, by Lemma 30 we have

$$\mathcal{C}_{cM}^{n+1} \bar{\Gamma}^{cM,cM,cM}(t,x;T,y) \leq 2\, \mathcal{C}_{cM}^{n+2} \bar{\Gamma}^{cM,cM,cM}(t,x;T,y),$$
$$\mathcal{C}_{cM} \bar{\Gamma}^{cM,cM,cM}(t,x;T,y) \leq \sqrt{n+2}\, \mathcal{C}_{cM}^{n+2} \bar{\Gamma}^{cM,cM,cM}(t,x;T,y).$$

Inserting the above results into (88) we obtain

$$\begin{aligned} |Z_{n+1}(t,x;T,y)| &\leq \frac{C^{n+1}C_0}{\sqrt{n!}} \frac{(T-t)^{\frac{n+1}{2}}}{\sqrt{n+1}} \left(\kappa + 2\|\lambda_1\|_\infty \left(\kappa + \sqrt{\tau}(1 + \|\lambda_1\|_\infty)\right) \mathcal{C}_{cM}^{n+2} \right) \bar{\Gamma}^{cM,cM,cM}(t,x;T,y) \\ &\leq \frac{C^{n+1}C_1 (T-t)^{\frac{n+1}{2}}}{\sqrt{(n+1)!}} \left(1 + \|\lambda_1\|_\infty \mathcal{C}_{cM}^{n+2}\right) \bar{\Gamma}^{cM,cM,cM}(t,x;T,y), \end{aligned}$$

where

$$C_1 = 2C_0 \left(\kappa + \sqrt{\tau}(1 + \|\lambda_1\|_\infty) \right).$$

Now, without loss of generality we can assume $C_1 \leq C$, and thus we obtain (84) for $n+1$. □

[7]Here Γ_E represents the Euler Gamma function.

We are now in position to prove Theorem 14 for $N = 1$. Indeed, by equations (60), (61) and Proposition 34 we have

$$\begin{aligned}
&|p(t,x;T,y) - p^{(1)}(t,x;T,y)| \\
&\leq \sum_{n=0}^{\infty} \frac{C^{n+1}}{\sqrt{n!}} \int_t^T (T-s)^{\frac{n}{2}} \int_{\mathbb{R}} p^{(0)}(t,x;s,\xi) \, \left(1 + \|\lambda_1\|_\infty \mathcal{C}_{cM}^{n+1}\right) \bar{\Gamma}^{cM,cM,cM}(s,\xi;T,y) d\xi ds
\end{aligned}$$

(and by Lemma 24 with $N = 0$ and $N = 1$ respectively)

$$\leq \sum_{n=0}^{\infty} \frac{C^{n+1}}{\sqrt{n!}} \int_t^T (T-s)^{\frac{n}{2}} \int_{\mathbb{R}} \bar{\Gamma}^{cM,cM,cM}(t,x;s,\xi) \, \left(1 + \|\lambda_1\|_\infty \mathcal{C}_{cM}^{n+1}\right) \bar{\Gamma}^{cM,cM,cM}(s,\xi;T,y) d\xi ds$$

(and by (87))

$$= 2(T-t) \left(\sum_{n=0}^{\infty} \frac{C^{n+1}(T-t)^{\frac{n}{2}}}{\sqrt{n!}} \left(1 + \|\lambda_1\|_\infty \mathcal{C}_{cM}^{n+1}\right) \bar{\Gamma}^{cM,cM,cM}(t,x;T,y) \right),$$

for any $x, y \in \mathbb{R}$ and $t, T \in \mathbb{R}$ with $0 < T - t \leq \tau$. Since

$$\sum_{n=0}^{\infty} \frac{C^{n+1}(T-t)^{\frac{n}{2}}}{\sqrt{n!}} \mathcal{C}_{cM}^{n+1} \bar{\Gamma}^{cM,cM,cM}(t,x;T,y)$$

can be easily checked to be convergent, this concludes the proof of Theorem 14 for $N = 1$. The proof for the general case is based on the same arguments. However, in the general case the technical details become significantly more complicated. Therefore, for sake of simplicity, we present here only a sketch with the main steps of the proof. We repeat the same iterative construction of $p(t,x;T,y)$, but using our N-th order approximation as a starting point. Thus, we replace $p^{(1)}(t,x;T,y)$ with $p^{(N)}(t,x;T,y)$ in (60), where Φ is now defined as

$$\Phi(t,x;T,y) = \sum_{n=0}^{\infty} Z_n^N(t,x;T,y),$$

where

$$\begin{aligned}
Z_0^N(t,x;T,y) &:= Lp^{(N)}(t,x;T,y), \\
Z_{n+1}^N(t,x;T,y) &:= \int_t^T \int_{\mathbb{R}} Lp^{(0)}(t,x;s,\xi) Z_n^N(s,\xi;T,y) d\xi ds.
\end{aligned}$$

Now, proceeding by induction, one can extend Propositions 32 and Proposition 33 to a general $n \in \mathbb{N}$. Eventually, after proving the identity

$$Lp^{(N)}(t,x;T,y) = \sum_{n=0}^{N} (L - L_n) v^{(N-n)}(t,x;T,y) + (L - L_0) v^{(1)}(t,x;T,y),$$

one will be able to prove the estimate (84) for $\left|Z_n^N(t,x;T,y)\right|$, from which Theorem 14 would follow exactly as in the case $N = 1$.

$\square$

D Proof of Equations (42) and (43)

Below, we will use the following repeatedly

$$\phi_n(\xi)e^{i\xi x} = a_n(\partial_x^2 - \partial_x)e^{i\xi x}, \qquad \phi_n'(\xi)e^{i\xi x} = ia_n(2\partial_x - 1)e^{i\xi x}, \qquad \phi_n''(\xi)e^{i\xi x} = -2a_n e^{i\xi x}.$$

Note that the above relations hold *only* when $\phi_n(\xi) = a_n(-\xi^2 - i\xi)$, as is the case when X is a diffusion with not jumps and no possibility of default. Now, using the above relations, we find

$$\begin{aligned}
u_1(t,x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi e^{i\xi x} \widehat{u}_1(t,\xi) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \int_0^t ds e^{i\xi x + (t-s)\phi_0(\xi)} (i\partial_\xi - \bar{x})\phi_1(\xi)\widehat{u}_0(s,\xi) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \int_0^t ds e^{i\xi x + (t-s)\phi_0(\xi)} (i\partial_\xi - \bar{x})\phi_1(\xi) e^{s\phi_0(\xi)} \widehat{h}(\xi) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \int_0^t ds \phi_1(\xi) e^{s\phi_0(\xi)} \widehat{h}(\xi) (-i\partial_\xi - \bar{x}) e^{i\xi x + (t-s)\phi_0(\xi)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \Big(t(x-\bar{x}) - \tfrac{1}{2} i t^2 \phi_0'(\xi) \Big) \phi_1(\xi) e^{i\xi x + t\phi_0(\xi)} \widehat{h}(\xi) \\
&= \Big(t(x-\bar{x}) + \tfrac{1}{2} t^2 a_0 (2\partial_x - 1) \Big) a_1(\partial_x^2 - \partial_x) u_0(t,x),
\end{aligned}$$

which establishes (42). Likewise, for u_2 we have

$$
\begin{aligned}
u_2(t,x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi e^{i\xi x} \widehat{u}_2(t,\xi) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \int_0^t ds e^{i\xi x + (t-s)\phi_0(\xi)} (i\partial_\xi - \bar{x})^2 \phi_2(\xi) \widehat{u}_0(s,\xi) \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \int_0^t ds e^{i\xi x + (t-s)\phi_0(\xi)} (i\partial_\xi - \bar{x}) \phi_1(\xi) \widehat{u}_1(s,\xi) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \int_0^t ds e^{i\xi x + (t-s)\phi_0(\xi)} (i\partial_\xi - \bar{x})^2 \phi_2(\xi) e^{s\phi_0(\xi)} \widehat{h}(\xi) \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \int_0^t ds \int_0^s dr e^{i\xi x + (t-s)\phi_0(\xi)} (i\partial_\xi - \bar{x}) \phi_1(\xi) e^{(s-r)\phi_0(\xi)} (i\partial_\xi - \bar{x}) \phi_1(\xi) e^{r\phi_0(\xi)} \widehat{h}(\xi) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \int_0^t ds \phi_2(\xi) e^{s\phi_0(\xi)} \widehat{h}(\xi) (-i\partial_\xi - \bar{x})^2 e^{i\xi x + (t-s)\phi_0(\xi)} \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \int_0^t ds \int_0^s dr \phi_1(\xi) e^{r\phi_0(\xi)} \widehat{h}(\xi) (-i\partial_\xi - \bar{x}) \phi_1(\xi) e^{(s-r)\phi_0(\xi)} (-i\partial_\xi - \bar{x}) e^{i\xi x + (t-s)\phi_0(\xi)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \left(t(x-\bar{x})^2 - it^2(x-\bar{x})\phi_0'(\xi) - \tfrac{1}{3}t^3(\phi_0'(\xi))^2 - \tfrac{1}{2}t^2\phi_0''(\xi) \right) \phi_2(\xi) e^{i\xi x + t\phi_0(\xi)} \widehat{h}(\xi) \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \Big(\tfrac{1}{2}t^2(x-\bar{x})^2\phi_1(\xi) - \tfrac{1}{2}it^3\,(x-\bar{x})\,\phi_1(\xi)\phi_0'(\xi) - \tfrac{1}{8}t^4\phi_1(\xi)(\phi_0'(\xi))^2 \\
&\qquad - \tfrac{1}{2}it^2\,(x-\bar{x})\,\phi_1'(\xi) - \tfrac{1}{6}t^3\phi_0'(\xi)\phi_1'(\xi) - \tfrac{1}{6}t^3\phi_1(\xi)\phi_0''(\xi) \Big) \phi_1(\xi) e^{i\xi x + t\phi_0(\xi)} \widehat{h}(\xi) \\
&= \left(t(x-\bar{x})^2 + t^2(x-\bar{x})a_0(2\partial_x - 1) + \tfrac{1}{3}t^3a_0^2(2\partial_x - 1)^2 + t^2a_0 \right) a_2(\partial_x^2 - \partial_x) u_0(t,x) \\
&\quad + \Big(\tfrac{1}{2}t^2(x-\bar{x})^2a_1(\partial_x^2 - \partial_x) + \tfrac{1}{2}t^3\,(x-\bar{x})\,a_1(\partial_x^2 - \partial_x)a_0(2\partial_x - 1) + \tfrac{1}{8}t^4a_1(\partial_x^2 - \partial_x)a_0^2(2\partial_x - 1)^2 \\
&\qquad + \tfrac{1}{2}t^2\,(x-\bar{x})\,a_1(2\partial_x - 1) + \tfrac{1}{6}t^3a_0(2\partial_x - 1)a_1(2\partial_x - 1) + \tfrac{1}{3}t^3a_1(\partial_x^2 - \partial_x)a_0 \Big) a_1(\partial_x^2 - \partial_x) u_0(t,x),
\end{aligned}
$$

which establishes (43).

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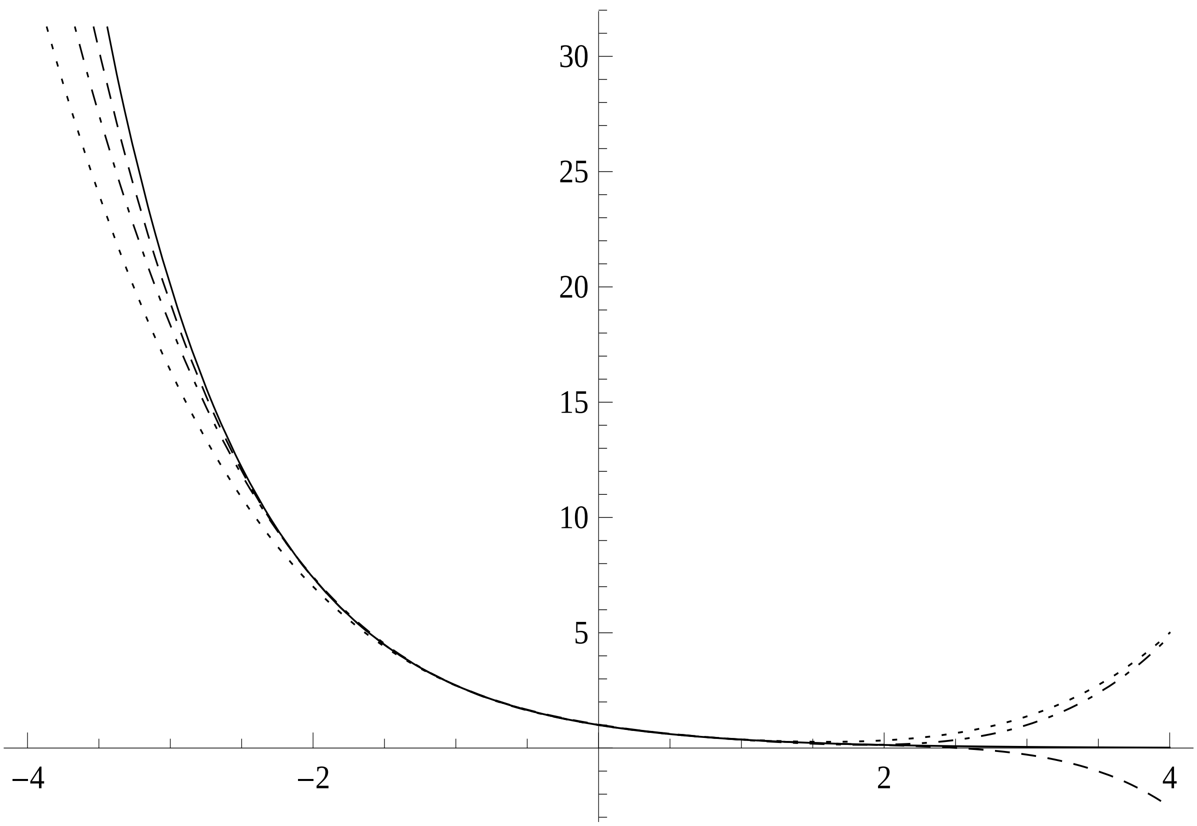

Figure 1: A comparison of the exact (solid), 2nd order two-point Taylor series approximation (dashed), 4th order Taylor series approximation (dotted), and 4th order Hermite polynomial expansion of the function $\exp(-x)$. For the two-point Taylor series approximation, we expand about $\bar{x}_1 = -1$ and $\bar{x}_2 = 1$. For the (usual) Taylor series and the Hermite polynomial approximations we expand about $\bar{x} = 0$.

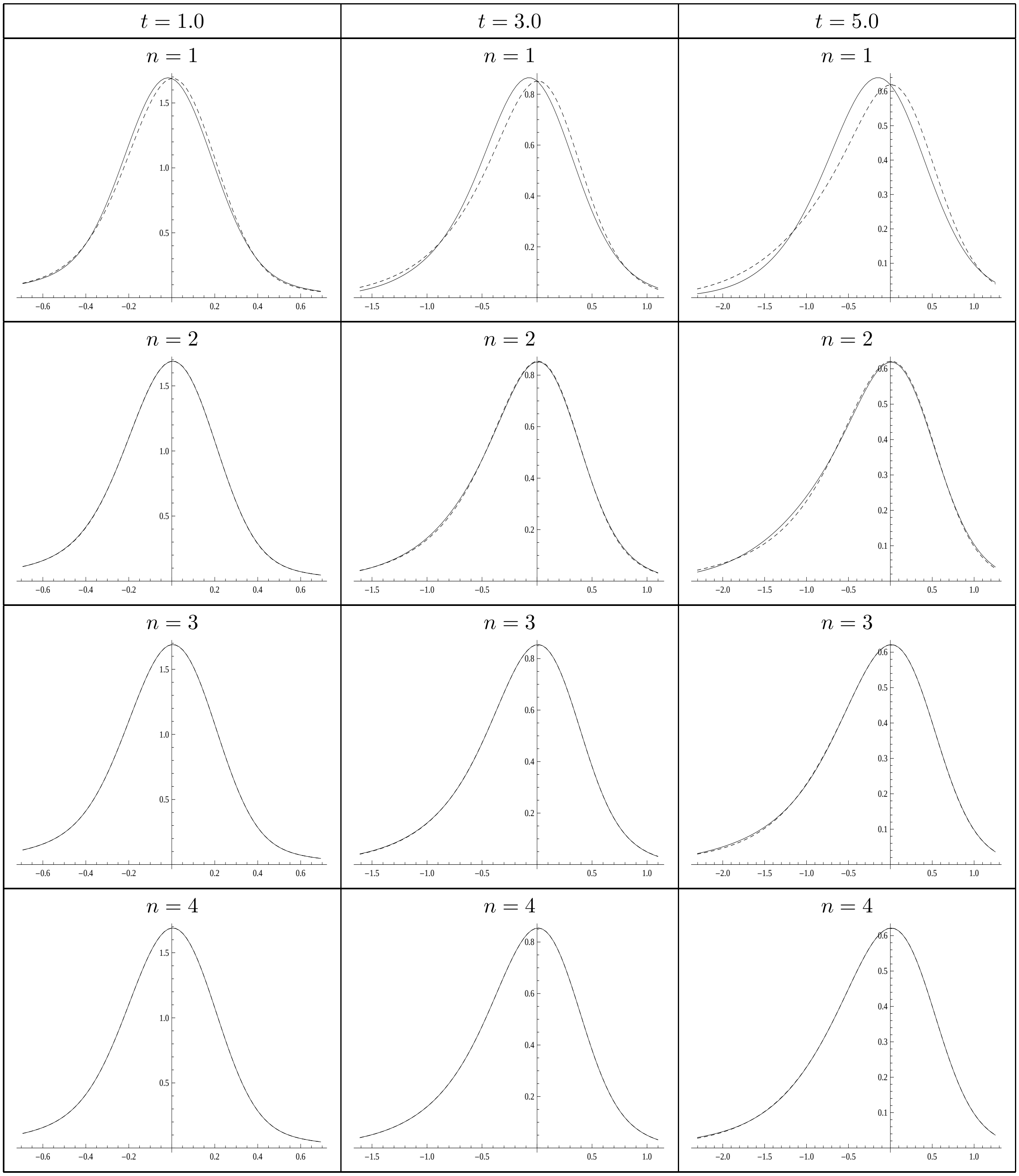

Figure 2: Using the model considered in Section 7.1.1 we plot $p^{(n)}(t,x;T,y)$ (solid black) and $p^{(n-1)}(t,x;T,y)$ (dashed black) as a function of y for $n = \{1,2,3,4\}$ and $t = \{1.0, 3.0, 5.0\}$ years. For all plots we use the Taylor series expansion of Example 5. Note that as n increases $p^{(n)}$ and $p^{(n-1)}$ become nearly indistinguishable. In these plots we set $\beta = 0.7$ All other parameter values are those listed in Section 7.1.1.

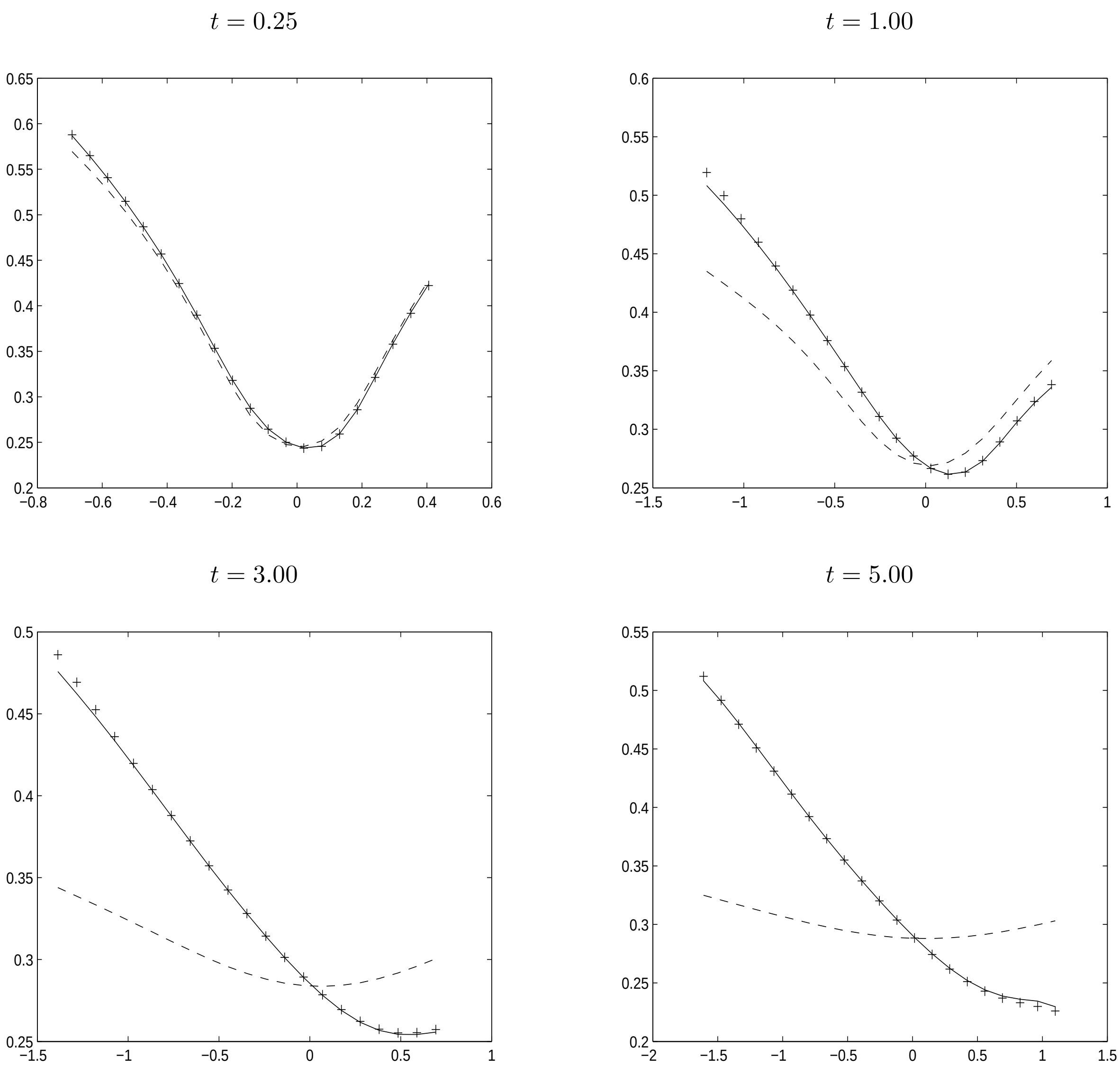

Figure 3: Implied volatility (IV) is plotted as a function of log-strike $k := \log K$ for the CEV-like model with Gaussian-type jumps of Section 7.1.1. The solid lines corresponds to the IV induced by $u^{(3)}(t,x)$, which is computed using the one-point Taylor expansion (see Example 6). The dashed lines corresponds to the IV induced by $u^{(0)}(t,x)$ (again, using the usual one-point Taylor series expansion). The crosses correspond to the IV induced by $u^{(MC)}(t,x)$, which is the price obtained from the Monte Carlo simulation.

t	k	$u^{(3)}$	u MC-95% c.i.	IV$[u^{(3)}]$	IV MC-95% c.i.
0.2500	-0.6931	0.0006	0.0006 - 0.0007	0.5864	0.5856 - 0.5901
	-0.4185	0.0024	0.0024 - 0.0025	0.4563	0.4553 - 0.4583
	-0.1438	0.0111	0.0110 - 0.0112	0.2875	0.2865 - 0.2883
	0.1308	0.1511	0.1508 - 0.1513	0.2595	0.2573 - 0.2608
	0.4055	0.5028	0.5024 - 0.5030	0.4238	0.4152 - 0.4288
1.0000	-1.2040	0.0009	0.0009 - 0.0010	0.5115	0.5176 - 0.5210
	-0.7297	0.0046	0.0047 - 0.0048	0.4174	0.4178 - 0.4199
	-0.2554	0.0314	0.0313 - 0.0316	0.3109	0.3102 - 0.3117
	0.2189	0.2781	0.2775 - 0.2784	0.2638	0.2620 - 0.2649
	0.6931	1.0034	1.0030 - 1.0041	0.3358	0.3296 - 0.3459
3.0000	-1.3863	0.0074	0.0081 - 0.0083	0.4758	0.4851 - 0.4870
	-0.8664	0.0224	0.0224 - 0.0227	0.4031	0.4029 - 0.4045
	-0.3466	0.0776	0.0773 - 0.0779	0.3280	0.3274 - 0.3288
	0.1733	0.3097	0.3094 - 0.3107	0.2690	0.2685 - 0.2703
	0.6931	1.0155	1.0150 - 1.0169	0.2558	0.2540 - 0.2604
5.0000	-1.6094	0.0160	0.0164 - 0.0166	0.5082	0.5111 - 0.5128
	-0.9324	0.0439	0.0436 - 0.0440	0.4118	0.4107 - 0.4121
	-0.2554	0.1504	0.1497 - 0.1507	0.3203	0.3194 - 0.3208
	0.4216	0.6139	0.6123 - 0.6142	0.2521	0.2500 - 0.2524
	1.0986	2.0050	2.0032 - 2.0057	0.2297	0.2163 - 0.2342

Table 1: Prices (u) and Implied volatility (IV$[u]$) as a function of time to maturity t and log-strike $k := \log K$ for the CEV-like model with Gaussian-type jumps of Section 7.1.1. The approximate price $u^{(3)}$ is computed using the (usual) one-point Taylor expansion (see Example 5. For comparison, we provide the 95% confidence intervals for prices and implied volatilities, which we obtain from the Monte Carlo simulation.

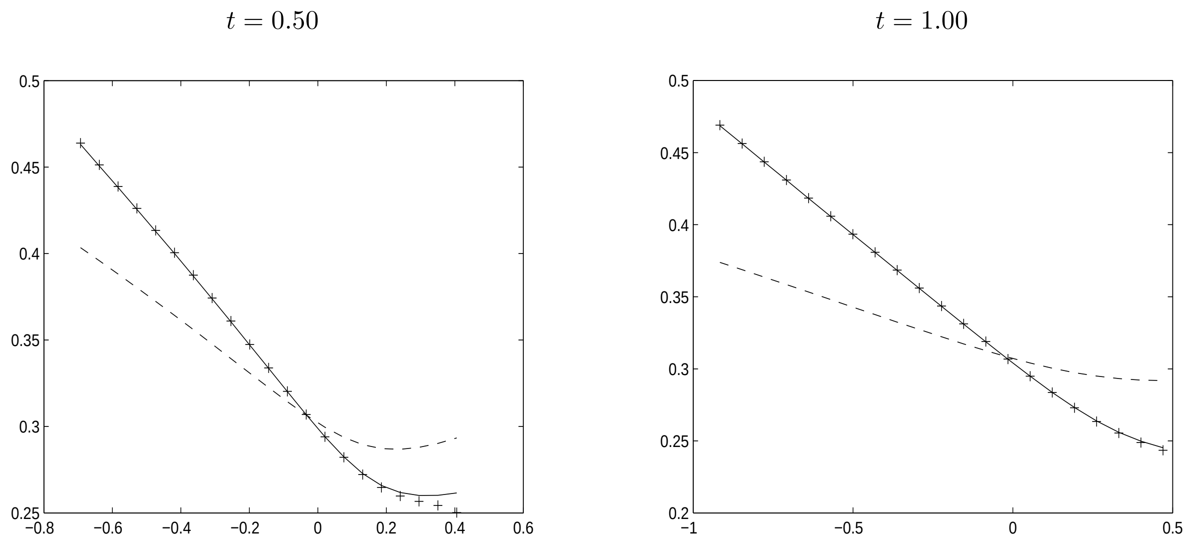

Figure 4: Implied volatility (IV) is plotted as a function of log-strike $k := \log K$ for the CEV-like model with Variance Gamma-type jumps of Section 7.1.2. The solid lines corresponds to the IV induced by $u^{(2)}(t,x)$, which is computed using the two-point Taylor expansion (see Example 6). The dashed lines corresponds to the IV induced by $u^{(0)}(t,x)$ (again, computed using the two-point Taylor series expansion). The crosses correspond to the IV induced by $u^{(MC)}(t,x)$, which is the price obtained from the Monte Carlo simulation.

t	k	$u^{(2)}$	u MC-95% c.i.	IV$[u^{(2)}]$	IV MC-95% c.i.
	-0.6931	0.0014	0.0014 - 0.0015	0.4631	0.4624 - 0.4652
	-0.4185	0.0070	0.0070 - 0.0071	0.4000	0.3995 - 0.4014
0.5000	-0.1438	0.0363	0.0362 - 0.0365	0.3336	0.3331 - 0.3346
	0.1308	0.1702	0.1697 - 0.1704	0.2727	0.2707 - 0.2736
	0.4055	0.5011	0.5004 - 0.5012	0.2615	0.2291 - 0.2646
	-0.9163	0.0028	0.0027 - 0.0028	0.4687	0.4678 - 0.4702
	-0.5697	0.0109	0.0109 - 0.0110	0.4057	0.4050 - 0.4068
1.0000	-0.2231	0.0473	0.0472 - 0.0476	0.3434	0.3428 - 0.3444
	0.1234	0.1970	0.1965 - 0.1974	0.2836	0.2825 - 0.2847
	0.4700	0.6033	0.6025 - 0.6037	0.2452	0.2355 - 0.2506

Table 2: Prices (u), Implied volatilities (IV$[u]$) and the corresponding confidence intervals from Figure 4.

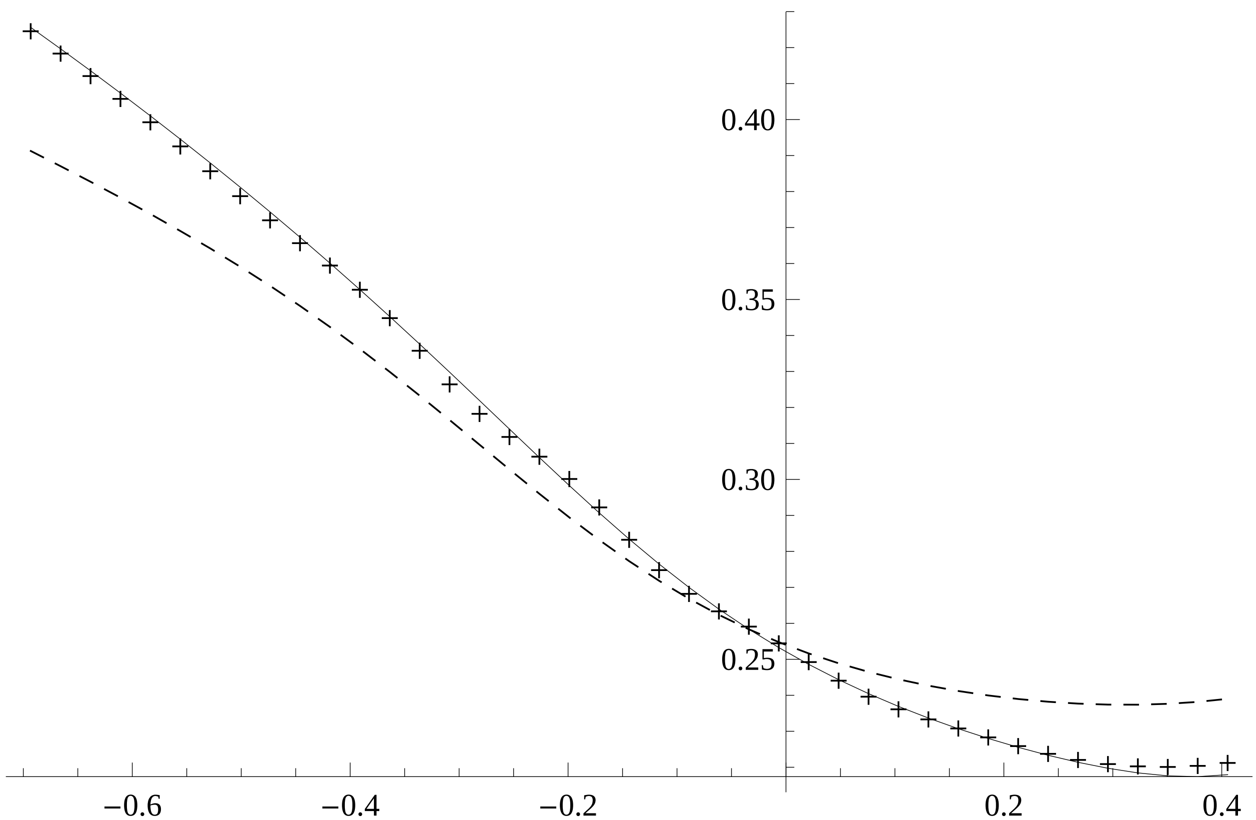

Figure 5: Implied volatility (IV) is plotted as a function of log-strike $k := \log K$ for the model of Section 7.2. The dashed line corresponds to the IV induced by $u^{(0)}(t,x)$. The solid line corresponds to the IV induced by $u^{(2)}(t,x)$. To compute $u^{(i)}(t,x)$, $i \in \{0,2\}$, we use the two-point Taylor series expansion of Example 6 The crosses correspond to the IV induced by the exact price, which is computed my truncating (50) at $n = 8$.

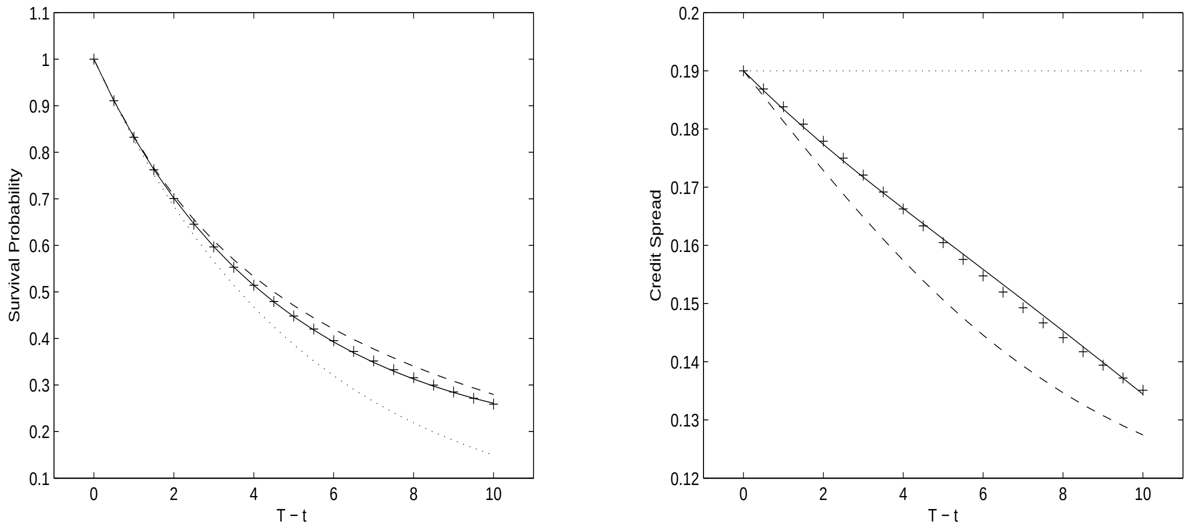

Figure 6: Left: survival probabilities $u(T-t,x) := \mathbb{Q}_x[\zeta > T|\zeta > t]$ for the JDCEV model described in Section 7.4. The dotted line, dashed line and solid line correspond to the approximations $u^{(0)}(T-t,x)$, $u^{(1)}(T-t,x)$ and $u^{(2)}(T-t,x)$ respectively, all of which are computed using Corollary 10. The crosses indicate the exact survival probability, computed by truncating equation (54) at $n = 70$. Right: the corresponding yields $Y^{(n)}(t,x;T) := -\log(u^{(n)}(T-t,x))/(T-t)$ on a defaultable bond. The parameters used in the plot are as follows: $x = \log(1)$, $\beta = -1/3$, $b = 0.01$, $c = 2$ and $a = 0.3$.

$T-t$	Y	$Y - Y^{(0)}$	$Y - Y^{(1)}$	$Y - Y^{(2)}$
1.0	0.1835	-0.0065	0.0022	0.0001
2.0	0.1777	-0.0123	0.0048	0.0003
3.0	0.1720	-0.0180	0.0071	0.0003
4.0	0.1663	-0.0237	0.0089	-0.0001
5.0	0.1605	-0.0295	0.0099	-0.0006
6.0	0.1548	-0.0352	0.0102	-0.0011
7.0	0.1493	-0.0407	0.0101	-0.0013
8.0	0.1442	-0.0458	0.0095	-0.0011
9.0	0.1394	-0.0506	0.0087	-0.0005
10.0	0.1351	-0.0549	0.0077	0.0007

Table 3: The yields $Y(t,x;T)$ on the defaultable bond described in Section 7.4: exact (Y) and nth order approximation ($Y^{(n)}$). We use the following parameters: $x = \log(1)$, $\beta = -1/3$, $b = 0.01$, $c = 2$ and $\delta = 0.3$.

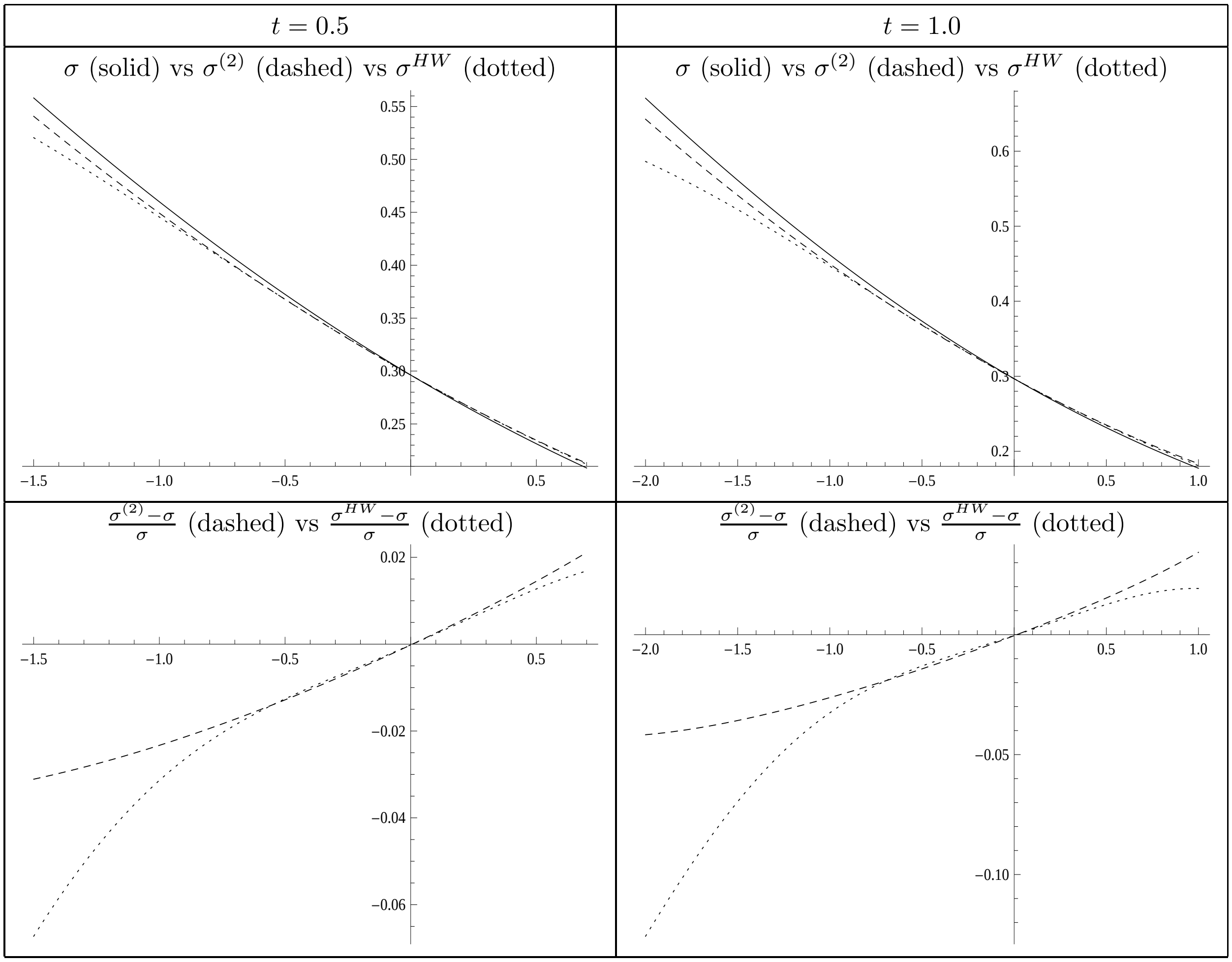

Figure 7: Top: implied volatility for the CEV model for maturities $t = 0.5$ and $t = 1.0$. The solid line is the exact implied volatility σ, computed by truncating Q in equation (55) at $n = 100$ terms and then inverting Black-Scholes numerically. The dashed line is the 2nd order implied volatility expansion $\sigma^{(2)}$, computed using equation (44). The dotted line is the Hagan-Woodward expansion of implied volatility σ^{HW}, computed using equation (56). Bottom: relative errors of the 2nd order (dashed) and Hagan-Woodward (dotted) implied volatility expansions. In all plots we use the following parameters: $\beta = 0.1$, $\delta = 0.3$, $X_0 = \log(1.0)$. Units of the horizontal axis are log-strike: $\log K$.

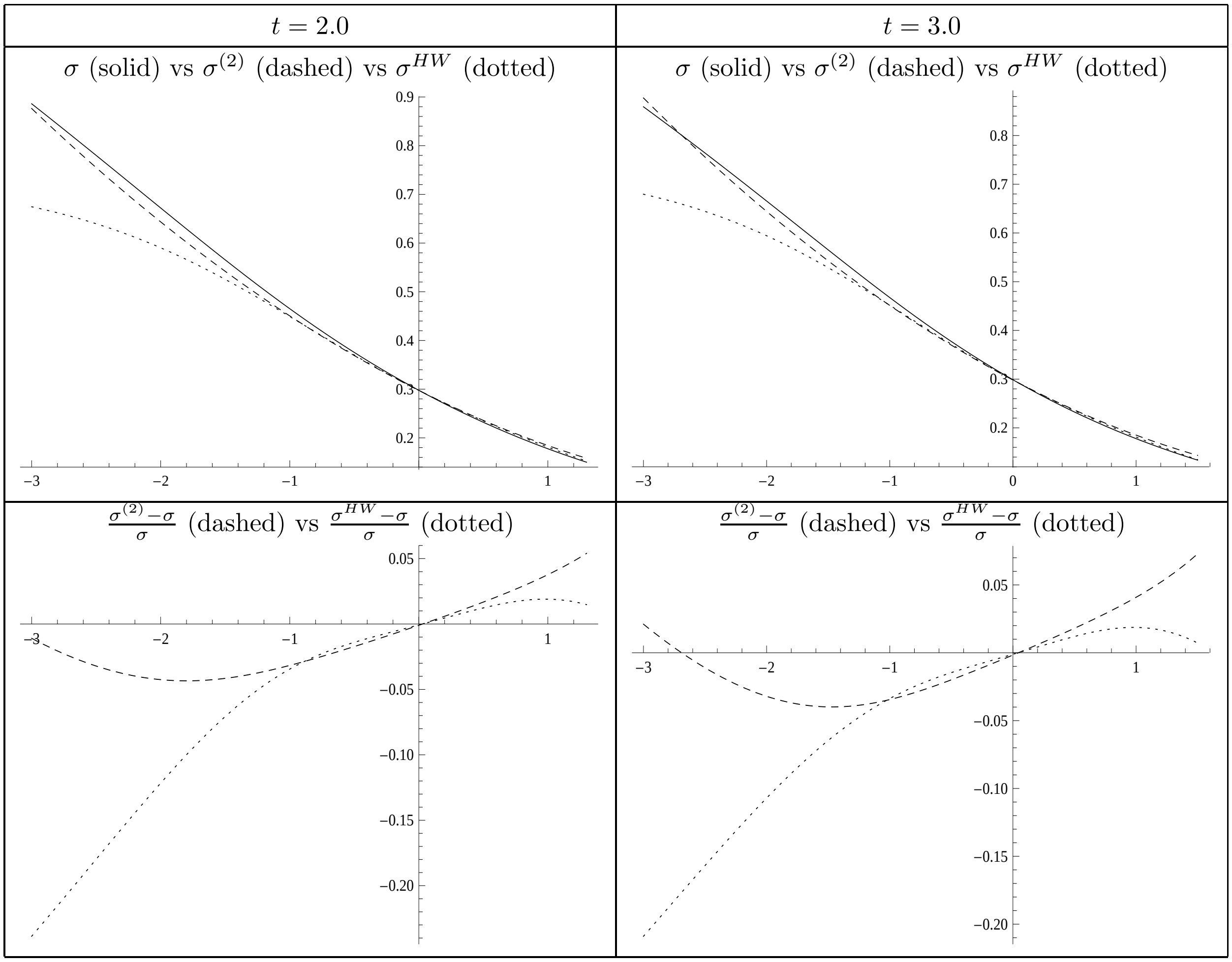

Figure 8: Top: implied volatility for the CEV model for maturities $t = 2.0$ and $t = 3.0$. The solid line is the exact implied volatility σ, computed by truncating Q in equation (55) at $n = 100$ terms and then inverting Black-Scholes numerically. The dashed line is the 2nd order implied volatility expansion $\sigma^{(2)}$, computed using equation (44). The dotted line is the Hagan-Woodward expansion of implied volatility σ^{HW}, computed using equation (56). Bottom: relative errors of the 2nd order (dashed) and Hagan-Woodward (dotted) implied volatility expansions. In all plots we use the following parameters: $\beta = 0.1$, $\delta = 0.3$, $X_0 = \log(1.0)$. Units of the horizontal axis are log-strike: $\log K$.